Non-gaussianity and cosmic uncertainty in curvaton-type models

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In curvaton-type models, observable non-gaussianity of the curvature perturbation would come from a contribution of the form $(\delta\sigma)^2$, where $\delta\sigma$ is gaussian. I analyse this situation allowing $\delta\sigma$ to be scale-dependent. The actual curvaton model is considered in more detail than before, including its cosmic uncertainty and anthropic status. The status of curvaton-type models after WMAP year three data is considered.

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I. INTRODUCTION

It is generally agreed that the primordial curvature perturbation ζ is caused by the perturbation of one or more scalar fields, those perturbations being generated on each scale at horizon exit during inflation. In curvaton-type models, a significant or dominant contribution to ζ is generated after slow-roll inflation ends, by a field σ whose potential is too flat to affect the inflationary dynamics.

The body of this paper is in three sections. Section II focuses on a perturbation of the form

$$\begin{aligned} \zeta(\mathbf{x}) &= \zeta_{\rm inf}(\mathbf{x}) + \zeta_\sigma(\mathbf{x}) \\ \zeta_\sigma(\mathbf{x}) &\equiv b\delta\sigma(\mathbf{x}) + \delta\sigma^2(\mathbf{x}). \end{aligned} \tag{1}$$

(The last term is written with the compact notation $\delta\sigma^2 \equiv (\delta\sigma)^2$, which will be used consistently.) Following [1], the spectrum, bispectrum and trispectrum of the perturbation are calculated, allowing for the first time spectral tilt in the spectrum of $\delta\sigma$.

In Section III, the generation of ζ is described using the δN formalism. In curvaton-type models ζ is given by Eq. (1) or its multi-field generalization. Non-gaussianity and scale-dependence are treated together, building on the separate discussions of [2, 3].

The prediction for ζ has cosmic uncertainty because it depends on the average value of the curvaton-like field in our part of the universe. One may assume that the probability distribution for this average typically is quite flat up to cutoff. The resulting probability distribution for ζ (the 'prior' for anthropic considerations) is model-dependent.

Section IV considers the actual curvaton model. A master formula for ζ is presented, including all known versions of the model. Assuming that the curvaton contribution dominates and that the curvaton has negligible evolution after inflation, the cosmic uncertainty and anthropic status of the curvaton model is described, extending the recent work of Garriga and Vilenkin [4] and Linde and Mukhanov [5]. Finally, in Section V we look at the status of curvaton-type, in the light of the recent measurement of negative spectral tilt for the curvature perturbation.

Standard material covered in for instance [6, 7] is taken for granted throughout, with fuller explanation given for more recent developments.

II. CALCULATING THE CORRELATORS

For convenience it is assumed that the spatial average of $\delta\sigma$ vanishes.

$$\overline{\delta\sigma} = 0. \tag{2}$$

This requirement is not essential,[1] because if Eq. (1) were valid with some $\overline{\delta\sigma} \neq 0$, one could arrive at $\overline{\delta\sigma} = 0$ by making the redefinitions

$$\begin{aligned} \delta\sigma &\to \delta\sigma - \overline{\delta\sigma} \\ b &\to b + 2\overline{\delta\sigma}. \end{aligned} \tag{3}$$

A. Working in a box

A generic cosmological perturbation, evaluated at some instant, will be denoted by by $g(\mathbf{x})$ and its Fourier components by

$$g_{\mathbf{k}} = \int d^3x\, e^{i\mathbf{k}\cdot\mathbf{x}}\, g(\mathbf{x}). \tag{4}$$

The integral goes over a box of size L, within which the stochastic properties are to be defined. To describe the cmb anisotropy the box should be much bigger than the size H_0^{-1} of the observable Universe.

Since the box introduces periodic boundary conditions, one requires that physically significant wavelengths are much shorter than the box size, corresponding to $kL \gg 1$. One can then regard the wave vector $\mathbf{k}$ as a continuous variable.

To describe the stochastic properties of cosmological perturbations within the box, one formally invokes an ensemble of universes and takes expectation values for observable quantities. The zero mode of each perturbation, corresponding to the spatial average within the

[1] In [1] and elsewhere, a contribution $-\overline{\delta\sigma^2}$ was added to ζ to make $\overline{\zeta} = 0$. That has no effect on the calculations, which deal only with Fourier modes of ζ with nonzero wavenumber.

box, is not regarded as a stochastic variable. The nonzero modes have zero expectation value, $\langle g_{\mathbf{k}}\rangle = 0$. (Both of these features are predicted by the inflationary cosmology.) It is usually supposed that the observable Universe corresponds to a typical member of the ensemble, so that the expectation values apply.

Since the stochastic properties are supposed to be invariant under translations and rotations (reflecting, within the inflationary paradigm, the invariance of the vacuum) a sampling of the ensemble in a given region may be regarded as a sampling of different locations for that region. One can say then, that within the box of size L we are dealing with the actual Universe, and that the expectation values refer to the location of the observable Universe within the box.

If $\ln(LH_0)$ is not exponentially large, it should be safe to assume that our location within the box is typical. On the other hand, the observable Universe may be part of a very large region around us with the same stochastic properties; a region so large that $\ln(LH_0)$ can be exponentially large. This is what happens within the inflationary cosmology, if inflation lasted for an exponentially large number of Hubble times before our Universe left the horizon. If such a super-large box is used, one should bear in mind the possibility that our location is untypical.

The spectrum $P_g(k)$ is defined by

$$\langle g_{\mathbf{k}}\, g_{\mathbf{k}'}\rangle = (2\pi)^3\delta^{(3)}(\mathbf{k}+\mathbf{k}')P_g(k). \quad (5)$$

It is useful to define $\mathcal{P}_g \equiv (k^3/2\pi^2)P_g$, also called the spectrum. After smoothing on a scale R, the variance is

$$\langle g^2(\mathbf{x})\rangle = \int_{L^{-1}}^{R^{-1}} \frac{dk}{k}\,\mathcal{P}_g(k). \quad (6)$$

The spectral index n_g and the spectral tilt t_g are defined as

$$t_g \equiv n_g - 1 \equiv \frac{d\ln g}{d\ln k}. \quad (7)$$

For constant tilt, $\mathcal{P}_g \propto k^{t_g}$.

If $\mathcal{P}_g(k)$ is sufficiently flat and the range of k is not too big,

$$\langle g^2\rangle \sim \mathcal{P}_g. \quad (8)$$

If instead it rises steeply, $\langle g^2\rangle \sim \mathcal{P}_g(R^{-1})$. In either case, the spectrum of a quantity is roughly its mean-square. This interpretation of the spectrum is implied in many discussions, including some in the present paper, but it should be applied with caution.

On cosmological scales $\mathcal{P}_\zeta$ is almost scale-invariant with $\mathcal{P}_\zeta^{1/2} = 5\times 10^{-5}$. At the 2σ level, the tilt is constrained by observation [8] to something like $t_\zeta = -0.03\pm 0.04$.

If the two-point correlator is the only (connected) one, the probability distribution of $g(\mathbf{x})$ is gaussian. Non-Gaussianity is signaled by additional connected correlators. Data are at present consistent with the hypothesis that ζ is perfectly gaussian, but they might not be in the future.

The bispectrum B_g is defined by

$$\langle g_{\mathbf{k}_1} g_{\mathbf{k}_2} g_{\mathbf{k}_3}\rangle = (2\pi)^3\delta^{(3)}(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3)B_g(k_1,k_2,k_3). \quad (9)$$

Instead of B_ζ it is more convenient to consider $f_{\rm NL}(k_1,k_2,k_3)$, defined by [9][2]

$$B_\zeta(k_1,k_2,k_3) = \frac{6}{5}f_{\rm NL}\left[P_\zeta(k_1)P_\zeta(k_2) + \text{cyclic}\right], \quad (10)$$

where the permutations are of $\{k_1,k_2,k_3\}$.

Current observation [12, 13] gives at 2σ level

$$-27 < f_{\rm NL} < 121. \quad (11)$$

Absent a detection, observation will eventually [14] bring this down to $|f_{\rm NL}| \lesssim 1$. At that level, the comparison of theory with observation will require second-order cosmological perturbation theory, whose development is just beginning [15].

The trispectrum T_g is defined in terms of the connected four-point correlator by as

$$\langle g_{\mathbf{k}_1} g_{\mathbf{k}_2} g_{\mathbf{k}_3} g_{\mathbf{k}_4}\rangle_c = (2\pi)^3\delta^{(3)}(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3+\mathbf{k}_4)T_g\,. \quad (12)$$

It is a function of six scalars, defining the quadrilateral formed by $\{\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3,\mathbf{k}_4\}$. It is convenient to consider $\tau_{\rm NL}$ defined by [1]

$$T_\zeta = \frac{1}{2}\tau_{\rm NL}P_\zeta(k_1)P_\zeta(k_2)P_\zeta(k_{14}) \,+\, 23\,\text{perms.}, \quad (13)$$

In this expression, $\mathbf{k}_{ij} \equiv \mathbf{k}_i + \mathbf{k}_j$, and the permutations are of $\{\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3,\mathbf{k}_4\}$ giving actually 12 distinct terms.

Current observation gives something like [16] $\tau_{\rm NL} \lesssim 10^4$, and absent a detection PLANCK data will give something like $\tau_{\rm NL} \lesssim 300$.

B. The correlators

If ζ is given by Eq. (1) its spectrum, bispectrum and trispectrum are given by the following expressions, with

[2] The sign and the prefactor make this definition coincide with the original one [11] in first-order cosmological perturbation theory, where $f_{\rm NL}$ was defined with respect to the Bardeen potential which was taken to be $\Phi = \frac{3}{5}\zeta$. (In many theoretical works, including previous works by the present author, $f_{\rm NL}$ is defined with the opposite sign.) At second order, which as we see later may be needed if $|f_{\rm NL}| \lesssim 1$, Φ and ζ are completely different functions and $f_{\rm NL}$ defined with respect to Φ has nothing to do with the $f_{\rm NL}$ of the present paper. Unfortunately, both definitions are in use at the second-order level.

all higher connected correlators vanishing;

$$\begin{aligned}
\mathcal{P}_\zeta &= \mathcal{P}_{\zeta_{\rm inf}} + \mathcal{P}_{\zeta_\sigma} && (14)\\
\mathcal{P}_{\zeta_\sigma} &= \mathcal{P}^{\rm linear}_{\zeta_\sigma} + \mathcal{P}^{\rm quad}_{\zeta_\sigma} && (15)\\
\mathcal{P}^{\rm linear}_{\zeta_\sigma} &= b^2\mathcal{P}_\sigma && (16)\\
\mathcal{P}^{\rm quad}_{\zeta_\sigma} &= \mathcal{P}_{\delta\sigma^2} = \frac{k^3}{2\pi}\int_{L^{-1}} d^3p\frac{\mathcal{P}_\sigma(p)\mathcal{P}_\sigma(|\mathbf{p}-\mathbf{k}|)}{p^3|\mathbf{p}-\mathbf{k}|^3} && (17)\\
B_\zeta &= B^{\rm linear}_{\zeta_\sigma} + B^{\rm quad}_{\zeta_\sigma} && (18)\\
B^{\rm linear}_{\zeta_\sigma} &= 8\pi^4 b^2\left(\frac{\mathcal{P}_\sigma(k_1)\mathcal{P}_\sigma(k_2)}{k_1^3k_2^3} + \text{cyclic}\right) && (19)\\
B^{\rm quad}_{\zeta_\sigma} &= B_{\delta\sigma^2}\\
&= (2\pi)^3\int_{L^{-1}} d^3p\frac{\mathcal{P}_\sigma(p)\mathcal{P}_\sigma(p_1)\mathcal{P}_\sigma(p_2)}{p^3p_1^3p_2^3} && (20)\\
T_\zeta &= T^{\rm linear}_{\zeta_\sigma} + T^{\rm quad}_{\zeta_\sigma} && (21)\\
T^{\rm linear}_{\zeta_\sigma} &= 8\pi^6 b^2\frac{P_\zeta(k_1)P_\zeta(k_2)P_\zeta(k_{14})}{k_1^3k_2^3k_{14}^3}\\
&+ \ 23\,\text{perms.} && (22)\\
T^{\rm quad}_{\zeta_\sigma} &= T_{\delta\sigma^2}\\
&= 4\pi^5\int_{L^{-1}} d^3p\frac{\mathcal{P}_\sigma(p)\mathcal{P}_\sigma(p_1)\mathcal{P}_\sigma(p_2)\mathcal{P}_\sigma(p_{24})}{p^3p_1^3p_2^3p_{24}^3}\\
&+ \ 23\,\text{perms.} && (23)
\end{aligned}$$

The 24 terms in Eq. (22) are actually 12 pairs of identical terms, and the 24 terms in Eq. (23) are actually 3 octuplets of identical terms. In the integrals $p_1 \equiv |\mathbf{p}-\mathbf{k}_1|$, $p_2 \equiv |\mathbf{p}+\mathbf{k}_2|$ and $p_{24} \equiv |\mathbf{p}+\mathbf{k}_{24}|$. The subscript L^{-1} indicates that the integrand is set equal to zero in a sphere of radius L^{-1} around each singularity. The integral (17) was given in [17], and the integrals (20) and (23) were given in [1] except that $\mathcal{P}_\sigma$ was taken to be scale-independent. The term $B^{\rm linear}_{\zeta_\sigma}$ was given in [11] and the term $T^{\rm linear}_{\zeta_\sigma}$ was given in [1, 18].

In the language of field theory, these expressions are obtained by contracting pairs of fields, after using the convolution

$$(\delta\sigma^2)_\mathbf{k} = \frac{1}{(2\pi)^3}\int d^3q\,\delta\sigma_\mathbf{q}\delta\sigma_{\mathbf{k}-\mathbf{q}}\,. \qquad (24)$$

The terms labeled 'quad' are generated purely by products of three $\delta\sigma^2$ terms, while the terms labeled 'linear' are generated by a product of two $b\delta\sigma$ terms and one $\delta\sigma^2$ term. Their evaluation is best done using Feynman-like graphs (cf. [19, 20]). The terms labeled 'linear' come from tree-level diagrams, while those labeled 'quad' come from closely-related one-loop diagrams.

Since $\delta\sigma$ is gaussian, its stochastic properties are determined entirely by its spectrum $\mathcal{P}_\sigma$. The correlators (hence all stochastic properties) of $\delta\sigma^2$ are also determined by $\mathcal{P}_\sigma$. By examining the large-p behaviour of Eqs. (17) and (20), we see that *the correlators of $\delta\sigma^2$ on a given scale are insensitive to the spectrum of $\delta\sigma$ on much smaller scales.*[3] In contrast with the case of quantum field theory, there is no divergence in the ultra-violet (large k) regime.

By examining the behaviour of Eqs. (17) and (20) near the singularities, we see that with sufficiently large positive tilt, the correlators of $\delta\sigma^2$ on a given scale are insensitive to the spectrum of $\delta\sigma$ also on much bigger scales. More will be said later about this infra-red regime.

The integral (17) can be evaluated exactly [17] to give

$$\mathcal{P}_{\delta\sigma^2}(k) = 4\mathcal{P}_\sigma^2\ln(kL). \qquad (25)$$

The integral (20) can be estimated as follows. Focusing on the singularity $p=0$, one can consider a sphere around it with radius k a bit less than $\min\{k_1,k_2\}$. The contribution from this sphere gives

$$B_{\delta\sigma^2} \simeq \frac{32\pi^4}{k_1^3k_2^3}\int_{L^{-1}}^k \frac{dp}{p} = \frac{32\pi^4}{k_1^3k_2^3}\ln(kL). \qquad (26)$$

A similar sphere around each of the other two singularities gives a similar contribution, and the contribution from these three spheres should be dominant because the integrand at large p goes like p^{-9}. (Applying this argument to the integral (17) happens to give exactly Eq. (25).) Evaluating $f_{\rm NL}$ we arrive at the estimate [1]

$$\begin{aligned}
\frac{3}{5}f_{\rm NL} &= b^2\frac{\mathcal{P}_\sigma^2}{\mathcal{P}_\zeta^2} + \frac{3}{5}f^{\rm quad}_{\rm NL} && (27)\\
\frac{3}{5}f^{\rm quad}_{\rm NL} &\simeq 4\frac{\mathcal{P}_\sigma^3}{\mathcal{P}_\zeta^2}\ln(kL) && (28)\\
&= \sqrt{\frac{1}{2\ln(kL)}}\left(\frac{\mathcal{P}_{\delta\sigma^2}}{\mathcal{P}_\zeta}\right)^{\frac{3}{2}}\mathcal{P}_\zeta^{-1/2} && (29)
\end{aligned}$$

with $k = \min\{k_1,k_2,k_3\}$. A similar estimate for the trispectrum gives [1]

$$\begin{aligned}
\tau_{\rm NL} &= 4b^2\frac{\mathcal{P}_\sigma^3}{\mathcal{P}_\zeta^3} + \tau^{\rm quad}_{\rm NL} && (30)\\
\tau^{\rm quad}_{\rm NL} &\simeq 16\frac{\mathcal{P}_\sigma^4}{\mathcal{P}_\zeta^3}\ln(kL) && (31)\\
&\simeq \frac{1}{\ln kL}\left(\frac{\mathcal{P}_{\delta\sigma^2}}{\mathcal{P}_\zeta}\right)^2\frac{1}{\mathcal{P}_\zeta}, && (32)
\end{aligned}$$

with $k = \min\{k_i, k_{jm}\}$.

It is easy to repeat these estimates for the case of constant nonzero tilt, $t_\sigma \equiv d\ln\mathcal{P}_\sigma/d\ln k$. To avoid rather cumbersome expressions, I give the result for the bispectrum only in the regime where the k_i have an approximate common value k, and the result for the trispectrum

[3] To be precise, Eqs. (17), (20), and (23) converge at $p \gg k$ if the tilt t_σ is below respectively 3/2, 2 and 9/4. These conditions are well satisfied within the inflationary paradigm, which makes $|t_\sigma|$ well below 1.

only in the regime where both k_i and k_{ij} have an approximate common value k. Then, the only effect of tilt is to replace $\ln(kL)$ by

$$y(kL) \equiv \int_{L^{-1}}^{k} \frac{dp}{p}\left(\frac{p}{k}\right)^{t_\sigma} = \frac{1-(kL)^{-t_\sigma}}{t_\sigma}. \quad (33)$$

The following limits apply

$$y(kL) = \begin{cases} \frac{1}{t_\sigma} & (t_\sigma \gg 1/\ln(kL)) \\ \ln(kL) & (|t_\sigma| \ll 1/\ln(kL)) \\ \frac{(kL)^{|t_\sigma|}}{|t_\sigma|} & (t_\sigma \ll -1/\ln(kL) \end{cases} . \quad (34)$$

The expressions for the correlators in terms of y are;

$$\mathcal{P}_{\zeta_\sigma} = b^2\mathcal{P}_\sigma + \mathcal{P}_{\delta\sigma^2} \quad (35)$$
$$\mathcal{P}_{\delta\sigma^2} \simeq 4y(kL)\mathcal{P}_\sigma^2 \quad (36)$$
$$\mathcal{P}_{\zeta_\sigma} \simeq \mathcal{P}_\sigma\left(b^2 + 4y\mathcal{P}_\sigma\right) \quad (37)$$
$$\frac{3}{5}f_{\rm NL} \simeq \frac{\mathcal{P}_\sigma^2}{\mathcal{P}_\zeta^2}\left(b^2 + 4y\mathcal{P}_\sigma\right) \quad (38)$$
$$\simeq b^2\frac{\mathcal{P}_\sigma^2}{\mathcal{P}_\zeta^2} + \frac{1}{2}\sqrt{\frac{1}{y(kL)}}\left(\frac{\mathcal{P}_{\delta\sigma^2}}{\mathcal{P}_\zeta}\right)^{\frac{3}{2}}\frac{1}{\mathcal{P}_\zeta^{1/2}} \quad (39)$$
$$\tau_{\rm NL} \simeq 4\frac{\mathcal{P}_\sigma^3}{\mathcal{P}_\zeta^3}\left(b^2 + 4y\mathcal{P}_\sigma\right) \quad (40)$$
$$\simeq 4b^2\frac{\mathcal{P}_\sigma^3}{\mathcal{P}_\zeta^3} + \frac{1}{y(kL)}\left(\frac{\mathcal{P}_{\delta\sigma^2}}{\mathcal{P}_\zeta}\right)^2\frac{1}{\mathcal{P}_\zeta}. \quad (41)$$

The tilt of ζ is given by

$$t_\zeta = \frac{t_{\zeta_{\rm inf}}\mathcal{P}_{\zeta_{\rm inf}} + b^2 t_\sigma\mathcal{P}_\sigma + t_{\delta\sigma^2}\mathcal{P}_{\delta\sigma^2}}{\mathcal{P}_\zeta}, \quad (42)$$

with

$$t_{\delta\sigma^2} = \begin{cases} 2t_\sigma & (t_\sigma \gg 1/\ln(kL)) \\ 1/\ln(kL) & (|t_\sigma| \ll 1/\ln(kL)) \\ t_\sigma & (t_\sigma \ll -1/\ln(kL) \end{cases} \quad (43)$$

For zero or negative tilt, increasing the box size has an ever-increasing effect on the correlators. For positive tilt we can use a maximal box such that $\ln(kL_{\rm max}) \gg 1/t_\sigma$ and $y = 1/t_\sigma$ are good approximations, giving

$$\mathcal{P}_{\delta\sigma^2} \simeq \frac{4}{t_\sigma}\mathcal{P}_\sigma^2 \quad (44)$$
$$\frac{3}{5}f_{\rm NL} = b^2\frac{\mathcal{P}_\sigma^2}{\mathcal{P}_\zeta^2} + \frac{1}{2}t_\sigma^{1/2}\left(\frac{\mathcal{P}_{\delta\sigma^2}}{\mathcal{P}_\zeta}\right)^{\frac{3}{2}}\frac{1}{\mathcal{P}_\zeta^{1/2}} \quad (45)$$
$$\tau_{\rm NL} = 4b^2\frac{\mathcal{P}_\sigma^3}{\mathcal{P}_\zeta^3} + t_\sigma\left(\frac{\mathcal{P}_{\delta\sigma^2}}{\mathcal{P}_\zeta}\right)^2\frac{1}{\mathcal{P}_\zeta}. \quad (46)$$

C. Working in a minimal box

To minimize the cosmic uncertainty of the correlators, one might wish to choose the box size to be as small as possible, consistent with the condition $LH_0 \gg 1$ which is required so that it can describe the whole observable Universe [17]. How big LH_0 has to be depends on the accuracy required for the calculation of observables, using the curvature perturbation as the initial condition. As the equations required for that calculation involve k^2 rather than k it may be reasonable to suppose that very roughly 1% accuracy will be obtained with $LH_0 \sim 10$ and 0.01% accuracy with $LH_0 \sim 100$. Even with the latter, the minimal box size $L_{\rm min}$ corresponds only to

$$\ln(L_{\rm min}H_0) \simeq 5. \quad (47)$$

The range of cosmological scales is usually taken to be only $\Delta\ln k \simeq 14$, going from the size $H_0^{-1} \sim 10^4\,$Mpc of the observable Universe, to the scale $10^{-2}\,$Mpc which encloses a mass of order $10^6 M_\odot$ and which corresponds to the first baryonic objects.

Cosmological scales therefore correspond to roughly

$$\ln(kL_{\rm min}) \sim 5 \text{ to } 20. \quad (48)$$

Let us see how things work out with the minimum box size. We saw that the dependence on the box size is through the function $y(kL)$ given by Eqs. (33) and (34). With the minimal box, this function is of order 1 on all cosmological scales, provided that $|t_\sigma| \lesssim 1/20$. This bound on t_σ is more or less demanded by observation if $\zeta_{\rm inf}$ is negligible, but it can be far exceeded if $\zeta_{\rm inf}$ dominates. In the latter case I will allow only positive tilt, since strong negative tilt looks unlikely as seen in Section III G. Then y is at most $1/t_\sigma$, and hence still roughly of order 1.

To go further with the minimal box, it will be enough to consider the two extreme cases, that the correlators are dominated by either their 'linear' or their 'quad' contributions. From Eqs. (37), (38), and (40) the former case corresponds to

$$\mathcal{P}_\sigma \ll b^2. \quad (49)$$

Given Eq. (8), this corresponds to the linear term of ζ_σ dominating, while the opposite case corresponds to the quadratic term of ζ_σ dominating.

If the linear term dominates and $\zeta_{\rm inf}$ is negligible,

$$\mathcal{P}_\zeta = b^2\mathcal{P}_\sigma \quad (50)$$
$$\frac{3}{5}f_{\rm NL} = b^{-2} \quad (51)$$
$$\tau_{\rm NL} = (36/25)f_{\rm NL}^2. \quad (52)$$

This is the usually-considered case. With a change of normalization one can write [11] $\zeta = \delta\sigma + \frac{3}{5}f_{\rm NL}\delta\sigma^2$.

If instead the quadratic term dominates ζ_σ, it cannot dominate ζ or there would be too much non-gaussianity. Indeed, given the interpretation (8), the non-gaussian fraction is

$$r_{\rm ng} \equiv \left(\frac{\mathcal{P}_{\delta\sigma^2}}{\mathcal{P}_\zeta}\right)^{1/2} = \left(\frac{\mathcal{P}_{\zeta_\sigma}}{\mathcal{P}_\zeta}\right)^{1/2}, \quad (53)$$

and [1]

$$r_{\rm ng} \sim \left(|f_{\rm NL}|\mathcal{P}_\zeta^{1/2}\right)^{\frac{1}{3}} \simeq \left(\tau_{\rm NL}\mathcal{P}_\zeta\right)^{\frac{1}{4}}. \tag{54}$$

The present bound $|f_{\rm NL}| < 121$ requires $r_{\rm ng} < 0.2$, but the present bound on $\tau_{\rm NL} < 10^4$ requires $r_{\rm ng} < 0.07$. We see that *if the quadratic term dominates, the present bound on the trispectrum is a stronger constraint than the one on the bispectrum.* Absent a detection, the post-COBE bounds on the bispectrum and trispectrum will lead to about the same constraint, $r_{\rm ng} \lesssim 0.04$.

With the quadratic term dominating, the present bound on $\tau_{\rm NL}$ gives $\mathcal{P}_{\zeta_\sigma}^{1/2} < 3\times 10^{-6}$. This, though, is on the fairly large cosmological scales probed by the cmb anisotropy. With positive tilt $\mathcal{P}_{\zeta_\sigma}(k_{\rm max})$ could be much bigger, even not far below 1 leading to black hole formation.

D. Running the box size

We have found that the stochastic properties depend on the size of the box in which Eqs. (1) and (2) are supposed to hold. This seems to be incompatible with a basic tenet of physics concerning the use of Fourier series, that the box size should be irrelevant if it is much bigger than the scale of interest.

This situation was discussed in [1] on the assumption that $\mathcal{P}_\sigma$ is scale-independent, and it is easy to extend the discussion to the case of an arbitrary $\mathcal{P}_\sigma(k)$. The crucial point is that $\overline{\delta\sigma}$ is supposed to vanish *within the chosen box* of size L. Let us imagine now that this box is within a much bigger box of fixed size M, and see how things vary if the size and location of the smaller box are allowed to vary. We have in mind that the small box will be a minimal one, and in the case of constant positive tilt the big box might be the maximal one satisfying $\ln(MH_0)t_\sigma \gg 1$. Defined in the big box, ζ has the form (1), with some coefficient b and with $\overline{\delta\sigma}=0$.

Focus first on a particular box with size L, and denote quantities evaluated inside this box by a subscript L. In general $\overline{\delta\sigma}_L$ will not vanish, and absorbing its expectation value into b using Eq. (3) we find

$$b_L = b + 2\overline{\delta\sigma}_L. \tag{55}$$

Now, instead of considering a particular small box, let its location vary so that $\overline{\delta\sigma}_L$ becomes the original perturbation $\delta\sigma$ smoothed on the scale L. Then

$$\begin{aligned}\langle b_L^2\rangle &= b^2 + 4\langle\overline{\delta\sigma^2}_L\rangle \\ \langle\overline{\delta\sigma^2}_L\rangle &= \int_{M^{-1}}^{L^{-1}} \frac{dk}{k}\mathcal{P}_\sigma(k),\end{aligned} \tag{56}$$

where the expectation values refer to the big box.

The operations of smoothing and taking the expectation value commute. Therefore, if $\mathcal{P}^L_{\delta\sigma^2}$, $B^L_{\delta\sigma^2}$ and $T^L_{\delta\sigma^2}$ are the spectrum, bispectrum and trispectrum defined within a particular small box of size L, we should have

$$\langle\mathcal{P}^L_{\delta\sigma^2}\rangle = \mathcal{P}_{\delta\sigma^2} \tag{57}$$

$$\langle B^L_{\delta\sigma^2}\rangle = B_{\delta\sigma^2} \tag{58}$$

$$\langle T^L_{\delta\sigma^2}\rangle = T_{\delta\sigma^2}, \tag{59}$$

where the right hand sides and the expectation values refer to the big box. One can verify this explicitly using Eqs. (15), (18), and (21). Indeed, these equations apply to a box of any size. The hierarchy $k \gg L^{-1} \gg M^{-1}$ allows one to evaluate the changes in the integrals induced by the change $M\to L$, and to verify (for any form of the spectrum) that this change is precisely compensated by the change $b^2 \to \langle b_L^2\rangle$.

III. THE INFLATIONARY PREDICTION

In this section we see how Eq. (1) and its generalizations may be predicted by inflation. We take the relevant scalar fields to be canonically normalized, and focus mostly on slow-roll inflation with Einstein gravity. The latter restriction is not very severe because a wide class of non-Einstein gravity theories can be transformed to an 'Einstein frame' [7].

A. The curvature perturbation

The curvature perturbation ζ is defined on the spacetime slicing of uniform energy density ρ through the metric [9, 40, 44, 48, 49]

$$g_{ij} = a^2(t)e^{2\zeta(\mathbf{x},t)}\gamma_{ij}(\mathbf{x}), \tag{60}$$

where γ has unit determinant. Within the inflationary cosmology, if the tensor perturbation is negligible, $\gamma_{ij} = \delta_{ij}$.

By virtue of the separate universe assumption, the threads of spacetime orthogonal to the uniform density slicing are practically comoving. As a result, a comoving volume element $\mathcal{V}$ is proportional to $a^3(\mathbf{x})$ where

$$a(\mathbf{x},t) \equiv a(t)e^{\zeta(\mathbf{x},t)}. \tag{61}$$

This means that $a(\mathbf{x},t)$ is a locally-defined scale factor.

An alternative definition of ζ refers to the slicing where the metric has the form $g_{ij} = a^2\hat\gamma_{ij}$ (with again $\hat\gamma$ having unit determinant). This is usually called the spatially flat slicing, which it is if the tensor perturbation is negligible. Linear cosmological perturbation theory gives ζ in terms of $\delta\rho$ on the spatially flat slicing,

$$\zeta = -H\frac{\delta\rho}{\dot\rho} = \frac{\delta\rho}{3(\rho+P)}. \tag{62}$$

If $f_{\rm NL}$ turns out to be of order 1 though, it will be necessary to go to second order and then the non-perturbative definition (60) becomes more useful [3, 53].

The linear expression (62) is convenient if the fluid is the sum of fluids, each with its own $P(\rho)$. Defining on flat slices the constants

$$\hat{\zeta}_i \equiv \frac{\delta\rho_i}{3(\rho_i + P_i)}, \tag{63}$$

one has

$$\zeta(t) = \frac{\sum(\rho_i + P_i)\hat{\zeta}_i}{(\rho + P)}. \tag{64}$$

By virtue of the separate universe assumption, the change in the energy $\rho\mathcal{V}$ within a given comoving volume element is equal to $-Pd\mathcal{V}$ with P is the pressure. This is equivalent to the continuity equation

$$\dot{\rho} = -3\frac{\dot{a}(\mathbf{x},t)}{a(\mathbf{x},t)}(\rho + P) \tag{65}$$

$$\frac{\dot{a}(\mathbf{x},t)}{a(\mathbf{x},t)} = \frac{\dot{a}}{a} + \dot{\zeta}. \tag{66}$$

Remembering that this holds on uniform-density slices, we see that ζ is conserved [23, 44, 48, 49, 50, 51, 52] during any era when P is a unique function of ρ. That is guaranteed during any era where there is practically complete radiation domination ($P = \rho/3$), matter domination ($P = 0$) or kination ($P = \rho$). The generation of ζ may take place during any other era.

B. Generating the curvature perturbation

The idea is that during inflation, the vacuum fluctuation of each light field becomes, a few Hubble times *after* horizon exit, a classical perturbation. (To keep the language simple I shall loosely say that the classical perturbation is present *at* horizon exit.) The observed curvature perturbation, present a few Hubble times *before* cosmological scales start to enter the horizon, is generated by one or more of these classical field perturbations.

Opportunities for generating ζ occur during any era when there is no relation $P(\rho)$. Generation was originally assumed to take place promptly at horizon exit in a single-component inflation model [21, 22, 23]. Then it was realised [24] that in a multi-component inflation model there will be continuous generation during inflation, but it was still assumed that the curvature perturbation achieves its final value by the end of inflation.

The term 'curvaton-type models' in this paper denotes models in which a significant contribution to the curvature perturbation is generated after the end of slow-roll inflation, by the perturbation in a field which has a negligible effect on inflation. If the curvaton-like contribution is completely dominant then the mechanism of inflation is irrelevant.

The curvaton model itself was the original proposal. In this model, the oscillating curvaton field leads to a second reheating, and the curvature perturbation is caused by the perturbation in the curvaton field [25, 26, 27, 28, 29].[4] Alternatives to the curvaton model, which still use a reheating, are to have the curvature perturbation generated by an inhomogeneity in any or all of the decay rate [34, 35], the mass [36] or the interaction rate [37] of the particles responsible for the reheating. In that case the reheating can be the first one (caused by the scalar field(s) responsible for the energy density during inflation) or alternatively the particle species causing the reheating can be a fermion [38]. Other opportunities for generating the curvature perturbation occur at the end of inflation [32], during preheating [33], and at a phase transition producing cosmic strings [39].

To describe the generation of the curvature perturbation I make two assumptions. First, that the evolution of perturbations on a given cosmological scale k can be described using an idealized universe, which is smooth on some scale a bit smaller than $1/k$. Second, that the local evolution of this idealized universe in the super-horizon regime can be taken to be that of some unperturbed universe.

The first assumption is routinely made in cosmology, on both super- and sub-horizon scales. The second assumption is the separate universe assumption [23, 40, 41], which amounts to the statement that the smoothed universe becomes locally isotropic and homogeneous when the smoothing scale is much bigger than the horizon. Given the content of the Universe at a particular epoch, it may be checked using cosmological perturbation theory, or else using the gradient expansion [40, 42, 43, 44] with the additional assumption of local isotropy. Local isotropy is more or less [45] guaranteed [46] by inflation. Independently of particular considerations, the separate universe assumption has to be valid on a scale a bit larger than H_0^{-1}, or the concept of an unperturbed FLRW Universe would make no sense. It will therefore be valid on all cosmological scales provided that all relevant scales in the early Universe are much smaller, which will usually be the case [47].

I assume slow-roll inflation, noting possible generalizations along the way. The light fields ϕ_i are defined as

[4] Earlier papers [30] considered the generic scenario, in which a light scalar field gives a negligible contribution to the energy density and the curvature perturbation during inflation, but a significant one at an unspecified later epoch. Such a scenario becomes a curvaton-type model if that epoch is before cosmological scales start to leave the horizon, otherwise it may be an axion-type model giving a cdm isocurvature perturbation. Among curvaton-type models, the curvaton model is the one which generates the curvature perturbation from the perturbation in the amplitude of the oscillating curvaton field. It was described in [25], and a formula equivalent to the estimate $\zeta \sim \delta\sigma/\sigma$ was given in [26]. In [27] the curvaton model was advocated as the dominant cause of the curvature perturbation and a precise calculation of $\mathcal{P}_\zeta$ was made allowing for significant radiation. In [28] a significant inflaton contribution was allowed. The first calculation of $f_{\rm NL}$ was given in [29]. The curvaton mechanism with a pre-big-bang instead of inflation was worked out in [31].

those which satisfy flatness conditions;

$$\epsilon_i \ll 1 \qquad |\eta_{ij}| \ll 1\,, \tag{67}$$

where

$$\epsilon_i \equiv \frac{1}{2} M_{\mathrm{P}}^2 \left(\frac{V_i}{V}\right)^2 \tag{68}$$

$$\eta_{ij} \equiv M_{\mathrm{P}}^2 \frac{V_{ij}}{V}, \tag{69}$$

with $V_i \equiv \partial V/\partial\phi_i$ and $V_{ij} \equiv \partial^2 V/\partial\phi_i\partial\phi_j$.

The exact field equation for each light field,

$$\ddot{\phi}_i + 3H\dot{\phi}_i + V_i = 0, \tag{70}$$

is supposed to be well-approximated by

$$3H\dot{\phi}_i = -V_i. \tag{71}$$

In these expressions H is the Hubble parameter, related to V by

$$3M_{\mathrm{P}}^2 H^2 = V + \frac{1}{2}\sum \dot{\phi}_i^2. \tag{72}$$

By virtue of Eqs. (67), (68), and (71), this becomes $3M_{\mathrm{P}}^2H^2 \simeq V$, and H is slowly-varying corresponding to almost-exponential inflation;

$$\left|\frac{1}{H^2}\frac{dH}{dt}\right| \simeq 2\epsilon \ll 1, \tag{73}$$

with

$$\epsilon \equiv \sum \epsilon_i. \tag{74}$$

(Despite the notation, it is $\sqrt{\epsilon_i}$ and not ϵ_i which transforms as a vector in field space.) If inflation is almost exponential but not necessarily slow-roll, it may still be useful to define the light fields by Eqs. (67)–(69), with V in the denominator replaced by $3M_{\mathrm{P}}^2H^2$.

Focusing on a given epoch during inflation, it may be convenient to choose the field basis so that one field ϕ points along the inflationary trajectory. I will call it the inflaton, which coincides with the standard terminology in the case of single-component inflation. I will denote the orthogonal light fields (assuming that they exist) by σ_i. In an obvious notation $\epsilon_{\sigma_i} = 0$ and $\epsilon = \epsilon_\phi$ initially. One may define also $\eta \equiv \eta_{\phi\phi}$.

From Eqs. (67)–(69) and (71), the gradient of the potential is slowly varying;

$$\frac{1}{H}\frac{dV_i}{dt} = -\eta_{ij}V_j. \tag{75}$$

The possible inflationary trajectories are the lines of steepest descent of the potential. The trajectories may be practically straight (single-component inflation) or significantly curved in the subspace of two or more light fields (multi-component inflation, called double inflation in the case of two fields). The terms single- and multi-component refer to the viewpoint that the inflaton field is a vector in field space. For single-component inflation, $\epsilon_{\sigma_i} \ll \epsilon$, and the inflaton field ϕ hardly changes direction in field space , so that one can choose a practically fixed basis $\{\phi, \sigma_i\}$.

C. The δN formula

To evaluate the curvature perturbation generated by the vacuum fluctuations of the light fields, we can use the δN formalism [2, 3, 24, 44, 54]. It gives $\zeta(\mathbf{x},t)$, in terms of the light field perturbations defined on a flat slice at some fixed 'initial' epoch during inflation;

$$\phi_i(\mathbf{x}) = \phi_i + \delta\phi_i(\mathbf{x}). \tag{76}$$

Keeping terms which are linear and quadratic in $\delta\phi_i$, the time-dependent curvature perturbation is

$$\begin{aligned}\zeta(\mathbf{x},t) &= \delta N(\phi_i(\mathbf{x}),\rho(t)) \\ &= \sum_i N_i\delta\phi_i(\mathbf{x}) + \frac{1}{2}\sum_{ij} N_{ij}\delta\phi_i\delta\phi_j \end{aligned} \tag{77}$$

Here, $N(\phi_i,\rho)$ is the number of e-folds, evaluated in an unperturbed universe, from the initial epoch to an epoch when the energy density ρ has a specified value. In the second line, $N_i \equiv \partial N/\partial\phi_i$ and $N_{ij} \equiv \partial^2 N/\partial\phi_i\partial\phi_j$, both evaluated on the unperturbed trajectory. In known cases the first two terms of this expansion in the field perturbations are enough.

The curvature perturbation $\zeta(\mathbf{x},t)$ is independent of the 'initial' epoch. At that epoch let us work in a basis (ϕ,σ_i) so that $N_i = (N_\phi, N_{\sigma_i})$. The contribution of the inflaton to ζ is time-independent because $\delta\phi$ just corresponds to a shift along the inflaton trajectory. It is given by

$$\zeta_{\mathrm{inf}} = N_\phi\delta\phi + \frac{1}{2}N_{\phi\phi}(\delta\phi)^2 \tag{78}$$

$$N_\phi = \frac{1}{M_{\mathrm{P}}^2}\frac{V}{V'} \tag{79}$$

$$N_{\phi\phi}/N_\phi^2 = \eta - 2\epsilon. \tag{80}$$

Redefining ϕ we learn that the non-gaussianity coming from $\delta\phi^2$ is negligible ($|f_{\mathrm{NL}}| \ll 1$).[5] We can therefore write

$$\zeta_{\mathrm{inf}} \simeq N_\phi\delta\phi. \tag{81}$$

The fields σ_i correspond to shifts orthogonal to the inflaton trajectory. The perturbations $\delta\sigma_i$ give initially no contribution to the curvature perturbation; in other words the σ_i are *initially* isocurvature perturbations. If the N_{σ_i} were evaluated at a final epoch just a few Hubble times after the initial one, they would be practically zero because the possible trajectories can be taken to be practically straight and parallel over such a short time. The crucial point is that the N_{σ_i} can subsequently grow.

[5] The three-point correlator of $\delta\phi$ also [20, 55] gives $|f_{\mathrm{NL}}| \ll 1$. A different proof that single-component inflation gives $|f_{\mathrm{NL}}| \ll 1$ was given earlier by Maldacena [9].

Such growth has nothing to do with the situation at horizon exit, in particular there is no expression for the N_{σ_i} in terms of the potential gradient analogous to Eqs. (79) and (80).

In any case, ζ settles down at some stage to a final time-independent value, which persists until cosmological scales start to enter the horizon and is constrained by observation. This value is the one that we studied in the previous section, focusing on Eq. (1) which is obviously a special case of Eq. (77).

Although we focused on slow-roll inflation, the basic formula (77) can describe the generation of ζ from the vacuum fluctuation for any model of almost-exponential inflation, which need have nothing to do with scalar fields or Einstein gravity. The only requirement is that the specified light fields ϕ_i, satisfying Eqs. (67)–(69) with V in the denominator replaced by $3M_{\rm P}^2H^2$, determine the local evolution of the energy density and pressure until the approach of horizon entry. Going further, the ϕ_i need not be the light fields themselves, but functions of them, evaluated at an 'initial' epoch which might be after the end of inflation.

D. The gaussian approximation

In the limit where the potential of the light fields is perfectly flat, inflation is exponential (de Sitter spacetime) with constant H. At least up to second order in cosmological perturbation theory [55], the perturbations in the light fields are then generated from the vacuum without back-reaction. Each perturbation is classical, and a few Hubble after horizon exit is practically time-independent with the spectrum [10] $(H/2\pi)^2$. Keeping a slow variation of H and finite values of the parameters ϵ_i and η_{ij}, the three-point correlator of the light field perturbations has been calculated by Seery and Lidsey [55], and its effect has been shown indeed to be negligible in some particular cases [20, 55, 56]. In the following we take the light fields to be gaussian at horizon exit.

Since ζ is almost gaussian it will presumably be a good approximation to keep only the first term of Eq. (77). Let us recall briefly the situation for that case. Focusing on a particular scale, we can take the initial epoch to be the epoch of horizon crossing. (The horizon-crossing trick.) Then

$$\zeta(\mathbf{k}) = \sum_i N_{i*}\delta\phi_{i*}(\mathbf{k}), \tag{82}$$

where a star denotes the epoch of horizon exit. Using the summation convention,

$$\mathcal{P}_\zeta(k) = (H_*/2\pi)^2 N_{i*}N_{i*}. \tag{83}$$

To evaluate the tilt from this equation one may use the slow-roll expressions

$$\frac{d}{d\ln k} = -\frac{M_{\rm P}^2}{V}V_i\frac{\partial}{\partial\phi_i} \tag{84}$$

$$N_iV_i = M_{\rm P}^{-2}V, \tag{85}$$

to give [2, 6]

$$t_\zeta = \left(2\frac{\eta_{nm}N_nN_m}{N_iN_i} - 2\epsilon - \frac{2}{M_{\rm P}^2N_iN_i}\right)_*, \tag{86}$$

with the right hand side evaluated at horizon exit.

At horizon exit, let us use the basis ϕ, σ_i. If the inflaton contribution $\delta\phi$ dominates one recovers the standard predictions;

$$\mathcal{P}_\zeta = \frac{1}{2M_{\rm P}^2\epsilon_*}\left(\frac{H_*}{2\pi}\right)^2 \tag{87}$$

$$t_\zeta \simeq t_\phi = (2\eta - 6\epsilon)_*. \tag{88}$$

In this case, the tensor fraction $\mathcal{P}_{\rm t}/\mathcal{P}_\zeta$ is $r = 16\epsilon$.

The contributions $\delta\sigma_i$ to $\mathcal{P}_\zeta$ are positive making [6] $r \leq 16\epsilon$. If one or more of them completely dominates, r will be too small to ever observe. If just one of them dominates (call it σ) dominates,

$$\mathcal{P}_\zeta = N_{\sigma*}^2\left(\frac{H_*}{2\pi}\right)^2 \tag{89}$$

$$t_\zeta \simeq t_\sigma = (2\eta_{\sigma\sigma} - 2\epsilon)_*. \tag{90}$$

E. Non-gaussianity in curvaton-type models

Now I consider the quadratic term of Eq. (77), allowing for the first time the possibility of significant scale dependence.

The horizon-crossing trick for evaluating scale dependence may not work when the quadratic term is included, because the convolution (24) involves a range of wavenumbers. It can still be used if the correlators of interest involve only a narrow range of scales $k_i \sim k$, and one just wishes to evaluate the dependence on the overall scale k. Otherwise, one should take the 'initial' epoch for Eq. (77) to be a fixed one, after all relevant scales have left the horizon.

To handle this situation I suppose that inflation is single-component, and that there is just one non-inflaton field σ which is going to be the curvaton-like field. (With small modification the discussion still applies to multi-component inflation models generating negligible non-gaussianity.) Then

$$\zeta = \zeta_{\rm inf} + \zeta_\sigma \tag{91}$$

$$\zeta_{\rm inf} \simeq N_\phi\delta\phi \tag{92}$$

$$\zeta_\sigma = N_\sigma\delta\sigma + \frac{1}{2}N_{\sigma\sigma}\delta\sigma^2. \tag{93}$$

Up to the normalization of $\delta\sigma$, Eq. (93) is the same as Eq. (1).

Since $\epsilon_\sigma \ll \epsilon$, the perturbations on flat slices satisfy [57]

$$\frac{1}{H}\frac{d\delta\phi}{dt} = -(\eta - 2\epsilon)\,\delta\phi \tag{94}$$

$$\frac{1}{H}\frac{d\delta\sigma}{dt} = -\eta_{\sigma\sigma}\delta\sigma, \tag{95}$$

The second equation is just the first-order perturbation of the unperturbed slow-roll equation $3H\dot{\sigma} = -V_\sigma$. That equation therefore applies locally.

The solutions are

$$\delta\phi = e^{-\int_{t_*}^{t} H(\eta-2\epsilon)dt}\delta\phi_* \tag{96}$$

$$\delta\sigma = e^{-\int_{t_*}^{t} H\eta_{\sigma\sigma}dt}\delta\sigma_*. \tag{97}$$

These are to be evaluated at a fixed t which will be the 'initial' epoch for use in Eqs. (92) and (93). Keeping only the linear terms and evaluating the spectrum, we recover Eqs. (88) and (90) for the spectral indices.

From now on the focus will be on the quadratic potential

$$V(\sigma) = \frac{1}{2}m_*^2\sigma^2. \tag{98}$$

Also H is supposed to be sufficiently slowly varying that

$$t_\sigma \simeq 2\eta_{\sigma\sigma} \equiv \frac{2m_*^2}{3H_*^3}, \tag{99}$$

where H_* is now the practically constant value of H during inflation without reference to a particular epoch.

As with any situation involving scalar fields in the early Universe, one has to remember that the effective potential $V(\sigma)$ can be affected by the values of other scalar fields and change with time. In particular, supergravity gives for a *generic* field during inflation $|\eta_{\sigma\sigma}| \sim 1$. This marginally violates the slow-roll condition and corresponds roughly to $m_* \sim H_*$ and $t_\sigma \sim 1$. The field which dominates ζ must have $|\eta_{\sigma\sigma}| \lesssim 0.01$. In any case, the mass m_* appearing in Eq. (98) will generally not be the true mass m, defined in the vacuum.

F. Maximum wavenumber and the 'initial' epoch

The classical curvature perturbation ζ is generated up to some maximum wavenumber $k_{\rm max}$. This maximum is generally taken to correspond to a scale $1/k_{\rm max}$ far below the shortest scale of direct cosmological interest discussed in Section II C. Nevertheless the value of $k_{\rm max}$ matters. It represents the shortest possible scale for the formation of primordial black holes[6] and the shortest scale on which matter density perturbations can exist. As we shall see, short-scale perturbations in the curvaton density contribute to its mean density, and hence indirectly to the magnitude of ζ which the curvaton model generates.

If ζ is created during inflation, $k_{\rm max}$ is the scale $k_{\rm e}$ leaving the horizon at the end of inflation. It is given by

$$\frac{k_{\rm e}}{H_0} = e^N, \tag{100}$$

where N is the number of e-folds of slow-roll inflation after the observable Universe with present size H_0^{-1} leaves the horizon. For a high inflation scale with continuous radiation domination afterward $N \simeq 60$. To make $k_{\rm e}^{-1}$ comparable with the shortest cosmological would require $N \simeq 14$ (see the discussion after Eq. (47)) which is hard to achieve. For this reason it seems to have been assumed in all previous discussions that $k_{\rm max}^{-1}$ will be far below cosmological scales.

That assumption is not justified if ζ is created by a curvaton-type mechanism long after inflation. After inflation, the perturbation $\delta\sigma$ in the curvaton-type field redshifts away on scales entering the horizon. Therefore, $k_{\rm max}$ is the scale entering the horizon when ζ is created. (See [17] for the same phenomenon in the axion case) This scale leaves the horizon long before the end of inflation. To handle that situation, I will still equate $k_{\rm max}$ with $k_{\rm e}$ given by Eq. (100), but define N as the number of e-folds of *relevant* inflation after the observable Universe leaves the horizon, 'relevant' meaning e-folds which produce perturbations on scales bigger than ${k_{\rm max}}^{-1}$. Demanding only that $k_{\rm max}^{-1}$ is below the shortest cosmological scale we can allow a range

$$14 < N \lesssim 60. \tag{101}$$

Taking the extreme values $N = 60$ and (say) $t_\sigma = 0.4$ gives $e^{Nt_\sigma} \sim 10^{11}$.

The end of relevant inflation is the appropriate 'initial' epoch for use in Eq. (93). The spectrum of $\sigma_{\rm e}$ is

$$\mathcal{P}_{\sigma_{\rm e}} = \left(\frac{H_*}{2\pi}\right)^2 \left(\frac{k}{k_{\rm e}}\right)^{t_\sigma} = \left(\frac{H_*}{2\pi}\right)^2 e^{-Nt_\sigma} \left(\frac{k}{H_0}\right)^{t_\sigma}, \tag{102}$$

The factor e^{Nt_σ} is of order 1 if $Nt_\sigma \lesssim 1$. This is more or less demanded by the observational bound on t_ζ if the curvature perturbation is dominated by the curvaton contribution. In the opposite case large tilt is allowed, making e^{Nt_σ} exponentially large as we saw earlier.

The unperturbed field at the end of inflation is

$$\sigma_{\rm e}^2 = \sigma^2 e^{-N_L t_\sigma} \tag{103}$$

$$N_L \equiv \ln(k_{\rm e}L), \tag{104}$$

where σ is the practically unperturbed value of the curvaton field within the box of size L when it leaves the horizon, and N_L is the number of e-folds of relevant inflation after the box leaves the horizon.

G. Cosmic uncertainty

If inflation lasts for enough e-folds before the observable Universe leaves the horizon, the stochastic formalism [59] allows one to calculate the probability distribution of σ, for a random location of our Universe. If our location is typical the actual value of σ^2 will be roughly $\langle\sigma^2\rangle$.

[6] See for instance references in [58], where the quantum regime $k > k_{\rm max}$ is also considered.

If the variation of H is so slow that it can be ignored, one arrives at a particularly simple probability distribution which is described in this subsection. Generalizations to allow for varying H are given for instance in [5, 74].

1. The Bunch-Davies case

In the case of constant positive tilt the result can be obtained from the formalism already presented. To do this, one works in a maximal box and assumes that the average value of σ rolls down to a practically zero value well before the observable Universe leaves the horizon. At any subsequent epoch, the local value $\sigma(\mathbf{x})$ has a gaussian probability distribution with variance

$$\langle \sigma^2(\mathbf{x}) \rangle = \left(\frac{H_*}{2\pi}\right)^2 \int_0^{aH_*} \frac{dk}{k} \left(\frac{k}{aH_*}\right)^{t_\sigma} \quad (105)$$

$$= \left(\frac{H_*}{2\pi}\right)^2 \frac{1}{t_\sigma} = \frac{3H_*^4}{8\pi^2 m_*^2}. \quad (106)$$

This is the case considered by Bunch and Davies [10, 21].

Working within a smaller box with size L (thought of as a minimal one), $\sigma(\mathbf{x})$ has a spatial average and a perturbation;

$$\sigma(\mathbf{x}) = \sigma_L + \delta\sigma_L(\mathbf{x}), \quad (107)$$

where the classical perturbation includes all wavenumbers $k < aH$. The mean square within that box is

$$\langle \sigma^2(\mathbf{x}) \rangle_L = \sigma_L^2 + \langle \delta\sigma_L^2 \rangle_L \quad (108)$$

$$= \sigma_L^2 + \int_{L^{-1}}^{aH_*} \frac{dk}{k} \mathcal{P}_\sigma. \quad (109)$$

For a random location of the small box (within the maximal box) each term of Eq. (107) has a gaussian distribution. Adding the two variances gives

$$\langle \sigma^2(\mathbf{x}) \rangle = \int_0^{L^{-1}} \frac{dk}{k} \mathcal{P}_\sigma + \int_{L^{-1}}^{aH_*} \frac{dk}{k} \mathcal{P}_\sigma, \quad (110)$$

which agrees with Eq. (105).

When the minimal box first leaves the horizon the perturbation $\delta\sigma$ is negligible. For a random location of the minimal box, the variance of the unperturbed value σ is then practically equal to the Bunch-Davies expression (106).

We were defining the curvature perturbation (93) within a minimal box, because that has general applicability. In the Bunch-Davies case we can instead use a maximal box, big enough to ensure the condition $\sigma = 0$ before the observable Universe leaves the horizon. In that case, N_σ may vanish leaving only the quadratic term of Eq. (93). This will happen if $\sigma \to -\sigma$ is a symmetry of the theory, and it happens anyway in the actual curvaton model because N_σ is then determined directly by the potential. Then the correlators are given by Eqs. (44)–(46), with $b = 0$ and $\mathcal{P}_{\delta\sigma^2} = \mathcal{P}_{\zeta_\sigma}$;

$$\mathcal{P}_{\zeta_\sigma} \simeq \frac{4}{t_\sigma} \mathcal{P}_\sigma^2 \quad (111)$$

$$\frac{3}{5} f_{\rm NL} = \frac{1}{2} t_\sigma^{1/2} \left(\frac{\mathcal{P}_{\zeta_\sigma}}{\mathcal{P}_\zeta}\right)^{\frac{3}{2}} \frac{1}{\mathcal{P}_\zeta^{1/2}} \quad (112)$$

$$\tau_{\rm NL} = t_\sigma \left(\frac{\mathcal{P}_{\zeta_\sigma}}{\mathcal{P}_\zeta}\right)^2 \frac{1}{\mathcal{P}_\zeta}. \quad (113)$$

2. The general case

In general, Eq. (98) will contain higher terms

$$V(\sigma) = \frac{1}{2} m_*^2 \sigma^2 + \lambda\sigma^4 + \sum_{d>4} \lambda_d \frac{\sigma^d}{M_{\rm P}^{d-4}}. \quad (114)$$

They will not affect the Bunch-Davies result provided that

$$\lambda \ll t_\sigma^2 \quad (115)$$

$$\lambda_d \left(\frac{\sigma}{M_{\rm P}}\right)^{d-4} \ll t_\sigma^2. \quad (116)$$

For an arbitrary potential $V(\sigma)$, assuming still that inflation with practically constant H lasts for long enough, the probability of finding σ in a given interval is [59, 60]

$$P(\sigma) d\sigma \propto \exp(-8\pi^2 V / 3H_*^4) d\sigma. \quad (117)$$

The potential is $V(\sigma)$ is evaluated with all other relevant fields fixed, with the convention that its minimum vanishes.

There are two simple cases. If $V(\sigma)$ increases until it becomes at least of order H_*^4, the probability distribution is more or less flat out to a value $\sigma_{\rm max}$, such that $V \sim 3H_*^4/8\pi^2$.[7] For the Bunch-Davies case one can take $\sigma_{\rm max}^2$ to be the variance $(H_*/2\pi)^2/t_\sigma$ of the gaussian probability distribution. In this case the tilt is positive and it can be strong, corresponding to the lightness condition $t_\sigma \ll 1$ being only marginally satisfied.

Instead, σ might be a PNGB with the potential

$$V(\sigma) = \Lambda^4 \cos^2(\pi\sigma/f). \quad (118)$$

If $\Lambda \ll H_*$ the probability distribution for σ is extremely flat within the fundamental interval $0 < \sigma < f$. The effective mass at the maximum and minimum of the potential is $m_* = \pi\Lambda^2/f$ giving the maximum tilt as

$$|t_\sigma| \simeq \frac{\Lambda^2}{H_*^2} \frac{\Lambda^2}{f^2}. \quad (119)$$

[7] This differs from the estimate of the typical value given in [61, 63, 64].

To have a reasonable probability for being at the maximum requires $\Lambda \ll H_*$, and to have weak self-coupling so that the semi-classical theory used here makes sense requires $\Lambda \ll f$. Judging by this example, strong negative tilt looks unlikely and is not considered in the present paper.

On the other hand, slight negative tilt consistent with observation is possible. The probabilities for being at the maximum and minimum of the potential are related by

$$\frac{P_{\rm max}}{P_{\rm min}} = \exp\left(-\frac{8\pi^2\Lambda^4}{3H_*^4}\right), \tag{120}$$

and one can have say $t_\sigma = -0.05$ while keeping this ratio not too far below 1.

It must be emphasized that the probability distribution Eq. (117) is attained only if the variation of H is negligible, on the timescale for the rolling down of σ towards its zero value. Whatever it is, the late-time probability distribution is not relevant for the inflaton in a single-component inflation model, since its value a given number of e-folds before the end of inflation is obtained by integrating the trajectory $3H\dot\phi = -V'$.

If eternal inflation takes place around some maximum of the potential, H will be practically constant during the eternal inflation and all light fields will attain the probability distribution (117) except for the one driving eternal inflation. When eternal inflation ends and slow-roll inflation begins, the fields orthogonal to the inflationary trajectory will have this probability distribution. In a single-component inflation model it will still apply when the observable Universe leaves the horizon, if the value of V then is not much lower than it was during eternal inflation. That may be the case if the potential has a suitable maximum, which is more likely than one might think [65]. The distribution (117) might also be attained [5] if eternal inflation occurs high up on the chaotic inflation potential $V \propto \phi^2$. A further possibility, not yet investigated, is that the distribution (117) is attained if eternal inflation occurs at a *maximum* of the potential with high V; possibly well-motivated [66, 67] realizations of that case would be Natural Inflation [68] along with its hybrid [69, 70] and multi-component [71] generalizations.

IV. THE CURVATON MODEL

A. The setup

The curvaton model [25, 26, 27, 28] (see also [31]) is a particular realization of Eq. (93). The curvaton field σ at some stage is oscillating harmonically about $\sigma = 0$ under the influence of a quadratic potential $V = \frac{1}{2}m^2\phi^2$, with energy density $\rho_\sigma \propto a^{-3}$. This stage begins at roughly the epoch

$$H \sim m. \tag{121}$$

This mass m in these equations is taken to be the true vacuum mass, the idea being that the effect of other fields on $V(\sigma)$ will have become negligible by this time. That will be more or less true if the effective mass up to that time has been $\lesssim H(t)$.

When the harmonic oscillation begins, ρ_σ is supposed to be negligible compared with the total $\rho = \rho_\sigma + \rho_{\rm inf}$. The component $\rho_{\rm inf}$ (defined as the difference between ρ and ρ_σ) roughly speaking originates from the decay of the inflaton but there is no need of that interpretation. The curvaton contribution to the curvature perturbation is at this stage supposed be negligible.

Eventually the harmonic oscillation decays. During at least some of the oscillation era, $\rho_{\rm inf}$ is supposed to be radiation-dominated so that $\rho_\sigma/\rho_{\rm inf}$ grows and with it ζ.[8]

Originally $\rho_{\rm inf}$ was supposed to be radiation-dominated during the whole oscillation era, but the model is not essentially altered if $\rho_{\rm inf}$ contains a significant contribution from matter. This matter might be the homogeneously oscillating inflaton field which decays only after the onset of the curvaton oscillation [61, 64, 91], non-relativistic curvaton particles [5], other non-relativistic particles which decay before the curvaton or any combination of these.

The lightness of the curvaton field can be ensured by taking it to be a PNGB with the potential (118). This mechanism can work whether or not there is supersymmetry, and is easier to implement for the curvaton than for the inflaton [27, 61, 75].

Several curvaton candidates exist which were proposed already for other reasons. Using such a candidate, one might connect the origin of the curvature perturbation with particle physics beyond the Standard Model, or even with observations at colliders and detectors. Among the candidates are a right-handed sneutrino [76, 77, 78, 79, 80], a modulus [28, 79] (which might be a string axion [61, 81]), a Peccei-Quinn field [63, 82] and an MSSM flat direction [79, 83]. The right-handed sneutrino possibility was actually discovered serendipitously [76] by authors who were unaware of the curvaton model, which shows that the curvaton model is not particularly contrived.

B. The master formula

In this subsection the basic approach is that of [27, 29], which works with the the first-order perturbation theory expression (62). This is applied after the onset of the harmonic oscillation of the curvaton, when the cosmic fluid has two components. The final value of ζ is taken to be the one evaluated just before the curvaton decays, which is taken to occur instantaneously on a slice

[8] Equations are derived on the assumption that these quantities are initially negligible. Presumably those same equations will provide a crude approximation even if the growth is negligible, due either to the curvaton decaying promptly [72, 73] or to $\rho_{\rm inf}$ containing a negligible radiation component.

of uniform energy density, at an epoch $H \sim \Gamma$ where Γ is the decay width.

Keeping this basic setup, the treatment of [27, 29] will be generalized to allow for several possible effects.[9] The inflaton component $\rho_{\rm inf}$ is not required to be purely radiation. The contribution of the inflaton perturbation to ζ is not required to be negligible. The tilt t_σ is taken into account. Attention will focus on constant tilt which is either small ($|t_\sigma| \lesssim 10^{-2}$, or else large and positive, which is allowed if the curvaton contribution to ζ is subdominant. In that case strong non-gaussianity will also be allowed.

Evolution of the curvaton field after the end of inflation will be allowed. One possibility [64] for such evolution is the large effective mass-squared $V_{\sigma\sigma} \sim \pm H^2(t)$ predicted by supergravity for a generic field during matter domination (though not [90] during radiation domination). A more drastic possibility is for σ to be a PNGB corresponding to the angular part of a complex field, whose radial part varies strongly [75, 84, 91, 92].

In the presence of evolution, the oscillation may initially be anharmonic, but after a few Hubble times the amplitude presumably will have decreased sufficiently that the oscillation is harmonic, making Eq. (121) an adequate approximation. (A detailed discussion for the analogous axion case is given in [17].)

The evolution is given by

$$\ddot\sigma + 3H(t)\dot\sigma + V_\sigma = 0. \tag{122}$$

Since the inflaton perturbation just corresponds to a shift in time, the 'separate universes' are practically identical until after the onset of the oscillation, and Eq. (122) holds locally at each position. Let us define the amplitude $\sigma_{\rm os}(\mathbf{x})$ at the start of the harmonic oscillation on a spacetime slice of uniform energy density. Then $\sigma_{\rm os}(\sigma_{\rm e}(\mathbf{x}))$ is a function only of $\sigma_{\rm e}$, the $\mathbf{x}$-dependence coming purely from the fact that $\sigma_{\rm e}(\mathbf{x})$ is not defined on such a slice. If V is quadratic (with a constant or slowly-varying mass) $\sigma_{\rm os}(\sigma_{\rm e})$ is practically linear.

Knowing $\sigma_{\rm os}(\mathbf{x})$ we can calculate[10]

$$\begin{aligned}\rho_\sigma(\mathbf{x}) &= \frac{m^2}{2}\left(\sigma_{\rm os} + \delta\sigma_{\rm os}(\mathbf{x})\right)^2 &(123)\\ \bar\rho_\sigma &= \frac{m^2}{2}\overline{\sigma^2_{\rm os}} &(124)\\ \overline{\sigma^2_{\rm os}} &= \sigma^2_{\rm os} + \langle\delta\sigma^2_{\rm os}\rangle &(125)\\ \delta\rho_\sigma &= \frac{m^2}{2}\left(2\sigma_{\rm os}\delta\sigma_{\rm os} + \delta\sigma^2_{\rm os}\right). &(126)\end{aligned}$$

To go further we expand $\sigma_{\rm os}(\sigma_{\rm e})$ to second order in $\delta\sigma_{\rm e}$ giving [84]

$$\delta\sigma_{\rm os}(\sigma_{\rm e}(\mathbf{x})) = \sigma'_{\rm os}\delta\sigma_{\rm e}(\mathbf{x}) + \frac12\sigma''_{\rm os}\delta\sigma^2_{\rm e}(\mathbf{x}), \tag{127}$$

and

$$\begin{aligned}\delta\rho_\sigma &= \frac{m^2\sigma^2_{\rm os}}{2}\left[2q\frac{\delta\sigma_{\rm e}}{\sigma_{\rm e}} + u\left(\frac{q\delta\sigma_{\rm e}}{\sigma_{\rm e}}\right)^2\right] &(128)\\ \bar\rho_\sigma &= \frac{m^2 p}{2}\sigma^2_{\rm os} &(129)\\ q &\equiv \sigma_{\rm e}\sigma'_{\rm os}/\sigma_{\rm os} &(130)\\ u &\equiv 1 + \sigma_{\rm os}\sigma''_{\rm os}/\sigma'^2_{\rm os} &(131)\\ p &\equiv \frac{\overline{\sigma^2_{\rm os}}}{\sigma^2_{\rm os}} = 1 + uq^2\frac{\langle\delta\sigma^2_{\rm e}\rangle}{\sigma^2_{\rm e}}. &(132)\end{aligned}$$

If $\delta\sigma$ has negligible evolution, or if $\sigma_{\rm os}(\sigma_{\rm e})$ is linear corresponding to a quadratic potential, then $q = u = 1$. It has been shown [93] that even slight anharmonicity could in certain cases give $|u| \ll 1$, and as we have seen strong evolution is also possible making both q and u very different from 1.

Except in Section IV F, all calculations will be done with the minimal box size, corresponding to $\ln(LH_0)$ not too far above 1. We need $\langle\delta\sigma^2_{\rm e}\rangle$;

$$\begin{aligned}\langle\delta\sigma^2_{\rm e}\rangle &\simeq \left(\frac{H_*}{2\pi}\right)^2\int_{L^{-1}}^{k_{\rm e}}\frac{dk}{k}\left(\frac{k}{k_{\rm e}}\right)^{t_\sigma} &(133)\\ &= \left(\frac{H_*}{2\pi}\right)^2 y(Lk_{\rm e}). &(134)\end{aligned}$$

This gives

$$p = 1 + uq^2\left(\frac{H_*}{2\pi\sigma}\right)^2 y(Lk_{\rm e})e^{N_L t_\sigma}. \tag{135}$$

with

$$y \simeq \min\begin{cases} N \\ 1/t_\sigma\end{cases} \tag{136}$$

A crude but usually adequate approximation is $\langle\delta\sigma^2_{\rm e}\rangle \simeq H^2$. (See [17] for a similar estimate of the axion perturbation.)

[9] Each effect was considered before, usually without any of the others. Strong tilt and non-gaussianity were considered in [5, 26]. The possible contribution of curvaton particles to $\rho_{\rm inf}$ was taken into account qualitatively in [5]. The Bunch-Davies case was considered in [5, 26, 78]. Curvaton evolution was partially taken into account in [29, 64, 84] (see also [3] for a treatment using the δN formalism)). The possible contribution of $\zeta_{\rm inf}$ was taken into account in [28, 61, 62]. All of this is at first order. The calculation to second order in cosmological perturbation theory was done in [87, 88] (re-derived in [3] using the δN formalism.) Also, the sudden-decay approximation was removed in [89], at first order only.

[10] Recall (footnote 2) that there is no need to demand $\overline{\delta\rho_\sigma} = 0$.

We have been evaluating ρ_σ and its perturbation at the beginning of the oscillation, on a slice where $\rho = \rho_{\rm inf}$ is uniform. The curvature perturbation is given by Eq. (64) in terms of the perturbations of the two fluids evaluated on the flat slicing. The curvaton perturbation on the uniform-density slicing is

$$\frac{\delta\rho_\sigma}{\overline{\rho}_\sigma} = \left(\frac{\delta\rho_\sigma}{\overline{\rho}_\sigma} - \frac{\delta\rho_{\rm inf}}{\rho_{\rm inf} + P_{\rm inf}} \right)_{\rm flat} . \qquad (137)$$

Each term in this expression is time-independent [29]. (Remember that the two-fluid description is only valid during the harmonic oscillation.) Using it, Eq. (64) evaluated just before the curvaton decay gives[11]

$$\zeta \;=\; \zeta_{\rm inf} + \zeta_\sigma \qquad (138)$$

$$\zeta_\sigma \;=\; f\frac{\delta\rho_\sigma}{\bar\rho_\sigma} \qquad (139)$$

$$3f \;=\; \frac{\bar\rho_\sigma}{\rho + P} = \frac{\bar\rho_\sigma}{\bar\rho_\sigma + \rho_{\rm inf} + P_{\rm inf}} \simeq \frac{\bar\rho_\sigma}{\rho} \equiv \Omega_\sigma. \qquad (140)$$

The approximation is adequate, because the 'sudden-decay' approximation has generally a significant error [89] in the regime $\Omega_\sigma < 1$.

The value of Ω_σ is to be calculated at the decay epoch $H \sim \Gamma$. It is sometimes convenient to write

$$\Gamma = \gamma m^3/M_{\rm P}^2. \qquad (141)$$

Then $\gamma \sim 10^{-2}$ corresponds to gravitational-strength decay [94] and one expects $\gamma \gtrsim 10^{-2}$.

Suppose first that there is continuous radiation domination during the oscillation. If $\Omega_\sigma \ll 1$, [27]

$$\Omega_\sigma \simeq \frac{\overline{\sigma_{\rm os}^2}}{M_{\rm P} m \gamma^{1/2}}. \qquad (142)$$

An approximation valid for any Ω_σ is therefore

$$\Omega_\sigma \;\simeq\; \frac{\overline{\sigma_{\rm os}^2}}{\overline{\sigma_{\rm os}^2} + C^2} \qquad (143)$$

$$C^2 \;=\; M_{\rm P} m\gamma^{1/2} = M_{\rm P}^2\sqrt{\frac{\Gamma}{m}}. \qquad (144)$$

Requiring a decay rate of at least gravitational strength, the first equality implies

$$C^2 \gtrsim 10^{-1} M_{\rm P} m. \qquad (145)$$

Requiring that the decay takes place before the onset of nucleosynthesis, corresponding to $\rho^{1/4} > 1\,{\rm MeV}$, the second equality implies

$$C^2 \gtrsim 10^{-21} M_{\rm P}^{5/2}/m^{1/2}. \qquad (146)$$

[11] According to the definitions made in this paper, $\hat\zeta_{\rm inf} = \zeta_{\rm inf}$, but $\hat\zeta_\sigma \neq \zeta_\sigma$.

These bounds cross at $m \sim 10^4\,{\rm GeV}$, implying $C \gtrsim 10^{11}\,{\rm GeV}$. It will be important later that C might be either bigger or smaller than H.

If $\rho_{\rm inf}$ has a matter component C is bigger. In particular, a contribution of curvaton particles, denoted by $\rho_{\rm c}$, gives

$$C^2 \simeq M_{\rm P} m \gamma^{1/2} + M_{\rm P}^2 \Omega_{\rm c}, \qquad (147)$$

where $\Omega_{\rm c}$ is evaluated at the onset of the oscillation. (The useful parameterization (143) was first given in [5], keeping just the contribution of curvaton particles.)

Using these equations we arrive at the master formula;

$$\zeta_\sigma \;\simeq\; \frac{2\Omega_\sigma}{3p}\left[q\frac{\delta\sigma_{\rm e}}{\sigma_{\rm e}} + \frac{u}{2}\left(q\frac{\delta\sigma_{\rm e}}{\sigma_{\rm e}}\right)^2 \right]. \qquad (148)$$

After adjusting the normalization of $\delta\sigma$ this has the form Eq. (1). Then the correlators are given by Eqs. (14)–(23).

C. Special cases

If we consider a single scale, Eqs. (35)–(41) apply, and if in addition this scale is taken to be $k \sim H_0 (\sim L^{-1})$ we can write things in terms of σ;

$$\zeta_\sigma \;\simeq\; \frac{2\Omega_\sigma}{3p}\left[q\frac{\delta\sigma}{\sigma} + \frac{u}{2}\left(q\frac{\delta\sigma}{\sigma}\right)^2 \right], \qquad (149)$$

with p given by Eq. (135). With that understanding let us evaluate the correlators in some special cases.

Let us assume that ζ_σ is dominated by the linear term, corresponding to

$$4\frac{q^2u^2}{\sigma^2}\left(\frac{H}{2\pi}\right)^2 \ll 1. \qquad (150)$$

Then

$$\mathcal{P}_{\zeta_\sigma}^{1/2} \;=\; \frac{2\Omega_\sigma q}{3p}\frac{H}{2\pi\sigma} \qquad (151)$$

$$\frac{3}{5}f_{\rm NL} \;=\; \frac{3pu}{4\Omega_\sigma}\left(\frac{\mathcal{P}_{\zeta_\sigma}}{\mathcal{P}_\zeta}\right)^2 \qquad (152)$$

$$\tau_{\rm NL} \;=\; (36/25) f_{\rm NL}^2 \left(\frac{\mathcal{P}_{\zeta_\sigma}}{\mathcal{P}_\zeta}\right)^3. \qquad (153)$$

In this case ζ_σ can be the dominant contribution, demanding $e^{Nt_\sigma} \sim 1$. Then, unless u is extremely small, $p \simeq 1$ leading to

$$\mathcal{P}_{\zeta}^{1/2} \;=\; \frac{2\Omega_\sigma q}{3}\frac{H}{2\pi\sigma} \qquad (154)$$

$$\frac{3}{5}f_{\rm NL} \;=\; \frac{3u}{4\Omega_\sigma} \qquad (155)$$

$$\tau_{\rm NL} \;=\; (36/25) f_{\rm NL}^2. \qquad (156)$$

For $q = u = 1$, corresponding to negligible evolution or evolution under a quadratic potential, Eqs. (154) and (155) reduce to the standard result [29];

$$\mathcal{P}_\zeta^{1/2} \simeq \frac{\Omega_\sigma H}{3\pi\sigma} \quad (157)$$

$$\frac{3}{5} f_{\rm NL} = \frac{3}{4}\frac{1}{\Omega_\sigma}, \quad (158)$$

the bound on $f_{\rm NL}$ requiring $\Omega_\sigma \gtrsim 10^{-2}$.

Supposing further that the evolution actually is negligible,

$$\Omega_\sigma = \frac{\sigma^2}{\sigma^2 + C^2}. \quad (159)$$

This is the simplest version of the curvaton model. It gives $H \gtrsim \mathcal{P}_\zeta^{1/2} C$, and then Eq. (146) gives [84] the bound

$$H \gtrsim 10^7 \,\text{GeV}. \quad (160)$$

D. The case $\Omega_\sigma = 1$

In the limiting case where Ω_σ is indistinguishable from 1 there is no need of the sudden-decay approximation. Before curvaton decay is appreciable, the curvaton-dominated cosmic fluid has $P = 0$ making ζ, and hence ζ_σ, a constant. The local value of σ provides the initial condition for the evolution of the separate universes, making them identical. As a result, ζ remains constant throughout and after the curvaton decay process.

To evaluate the non-gaussianity in this case one can use the δN formalism, which for the curvaton model is equivalent to using second-order cosmological perturbation theory. Adopting the small-tilt and $p = 1$ assumptions, the calculation described in [3] applies so that

$$\frac{3}{5} f_{\rm NL} = \frac{3}{4} u - \frac{3}{2}. \quad (161)$$

With negligible evolution or a purely quadratic potential, $f_{\rm NL} = -\frac{5}{4}$. It would be interesting to know if such a small value will ever be observable.

E. Induced isocurvature perturbations

The status of isocurvature perturbations in the curvaton model is considered elsewhere [29, 85] on the assumption that $\rho_{\rm inf}$ contains no curvaton particles. Let us briefly reconsider the situation when that assumption is relaxed.

As defined by astronomers, an isocurvature perturbation S may be present in any or all of the baryon, cold dark matter or neutrino components of the cosmic fluid when cosmological scales first approach the horizon, being the fractional perturbation in the relevant number density on a slice of uniform energy density. Given the separate universe assumption, the inflaton perturbation $\delta\phi$ cannot generate an isocurvature perturbation since it just corresponds to a shift back and forth along the inflaton trajectory. Any orthogonal light field σ_i might create an isocurvature perturbation, and the same field might give the dominant contribution to the curvature perturbation so that the two perturbations would be fully correlated. This has been called a residual isocurvature perturbation [29, 47, 85].

In the curvaton model, a residual isocurvature perturbation obviously cannot be created after the curvaton decays. If the cdm or baryon number is is created *by* the curvaton decay and $\rho_{\rm inf}$ contains no curvaton particles, the argument of [29] gives a residual isocurvature perturbation $S \simeq -3(1 - \Omega_\sigma)\zeta$. This is viable only if Ω_σ is close to 1. Repeating the argument of [29] for the case that $\rho_{\rm inf}$ contains curvaton particles, one easily sees that they should be discounted when evaluating Ω_σ in the expression for S, allowing a true $\Omega_\sigma \ll 1$.

Finally, suppose that the cdm or baryon number is created before curvaton decay. Then, whether or not $\rho_{\rm inf}$ contains curvaton particles, the argument of [29] gives a residual isocurvature perturbation $S/\zeta \simeq -3(1 - \Omega_\sigma^{\rm crea})$ where the superscript denotes the epoch of creation.

The above formulas apply also to the fractional isocurvature perturbation in the lepton number density, from which the neutrino isocurvature perturbation can be calculated [29]. It is not out of the question to generate the big lepton number density, that is needed to give a significant neutrino isocurvature perturbation [86].

F. The Bunch-Davies case

So far σ is unspecified. This is the unperturbed value of the curvaton field when the minimal box leaves the horizon. Now we consider the case that σ^2 is equal to the variance $(H/2\pi)^2/t_\sigma$ of the Bunch-Davies distribution. As we saw earlier, it becomes simpler in that case to use a maximal box, such that $\sigma_{\rm e} = 0$ and

$$\overline{\sigma_{\rm e}^2} = \langle \delta\sigma_{\rm e}^2 \rangle = \left(\frac{H_*}{2\pi}\right)^2 \frac{1}{t_\sigma}. \quad (162)$$

Then the master formula can be written

$$\zeta_\sigma = \frac{1}{3}\Omega_\sigma u q^2 \hat{p} \frac{\delta\sigma_{\rm e}^2}{\langle \delta\sigma_{\rm e}^2 \rangle} \quad (163)$$

$$\hat{p} \equiv \frac{\langle \delta\sigma_{\rm e}^2 \rangle}{\overline{\sigma_{\rm os}^2}} = \frac{\overline{\sigma_{\rm e}^2}}{\overline{\sigma_{\rm os}^2}}. \quad (164)$$

In the following I set equal to 1 the evolution factor $qu\hat{p}$.

After adjusting the normalization of $\delta\sigma$, Eq. (111) gives

$$\mathcal{P}_{\zeta_\sigma}^{1/2} = \frac{2}{3}\Omega_\sigma t_\sigma^{1/2} e^{-N t_\sigma} \left(\frac{k}{H_0}\right)^{t_\sigma}. \quad (165)$$

If $\mathcal{P}_\zeta = \mathcal{P}_{\zeta_\sigma}$, observation requires $Nt_\sigma \lesssim 1$ and Eqs. (111)–(113) give

$$\mathcal{P}_\zeta^{1/2} \simeq \frac{2}{3}\Omega_\sigma\sqrt{t_\sigma} \quad (166)$$

$$\frac{3}{5}f_{\rm NL} \simeq \frac{3}{4\Omega_\sigma} \quad (167)$$

$$\tau_{\rm NL} = (36/25){f_{\rm NL}}^2. \quad (168)$$

In this case $t_\sigma \sim 10^{-9}/f_{\rm NL}^2$, which has to be very small indeed as was first noticed by Postma [78].

If $\mathcal{P}_{\zeta_\sigma} \ll \mathcal{P}_\zeta$, strong tilt is allowed. Then Eqs. (112) and (113) give

$$\frac{3}{5}f_{\rm NL} = 4t_\sigma^2\left(\frac{\Omega_\sigma}{3}\right)^3 e^{-3Nt_\sigma}\left(\frac{k}{H_0}\right)^{3t_\sigma}\mathcal{P}_\zeta^{-2} \quad (169)$$

$$\tau_{\rm NL} = 16t_\sigma^3\left(\frac{\Omega_\sigma}{3}\right)^4 e^{-4Nt_\sigma}\left(\frac{k}{H_0}\right)^{4t_\sigma}\mathcal{P}_\zeta^{-3}. \quad (170)$$

The present bound $f_{\rm NL} < 121$ requires $\Omega_\sigma e^{-Nt_\sigma} < 9\times 10^{-6}$ and the present bound $\tau_{\rm NL} < 10^4$ requires the stronger bound $\Omega_\sigma e^{-Nt_\sigma} < 2\times 10^{-6}$. These bounds are an extension of Eq. (54), derived now for a maximal box. As in that case, they apply only on large cosmological scales, allowing a large curvature perturbation n much smaller scales.

To summarize, we find in the Bunch-Davies case strong non-gaussianity on large scales ($k \sim H_0$) provided that $Nt_\sigma \gtrsim 1$. This is because a typical region of size H_0^{-1} then becomes very inhomogeneous by the end of relevant inflation. The non-gaussianity is reduced as the scale is decreased, and (with $\Omega_\sigma = 1$) is small on the scale leaving the horizon at the end of relevant inflation precisely because a typical region of size $k_{\rm e}^{-1}$ is still quite homogeneous.

Although the maximal box provides the neatest result, it is interesting also to see what happens with a minimal box. For a box of any size, Eq. (163) becomes

$$\zeta_\sigma = \frac{1}{3}\Omega_\sigma u q^2\hat{p}\langle\delta\sigma_{\rm e}^2\rangle^{-1}\left(2\sigma_{\rm e}\delta\sigma_{\rm e} + \delta\sigma_{\rm e}^2\right). \quad (171)$$

Setting $u = q = \hat{p} = 1$ and adjusting for the normalization of $\delta\sigma$, we can calculate the correlators of ζ_σ from Eqs. (37), (38), and (40) and then take their expectation values within a maximal box, to find

$$\langle\mathcal{P}_{\zeta_\sigma}\rangle = 4t_\sigma\left(\frac{\Omega_\sigma}{3}\right)^2 e^{-2Nt_\sigma}\left(\frac{k}{H_0}\right)^{t_\sigma} f \quad (172)$$

$$\frac{3}{5}\langle f_{\rm NL}\rangle = 4t_\sigma^2\left(\frac{\Omega_\sigma}{3}\right)^3 e^{-3Nt_\sigma}\left(\frac{k}{H_0}\right)^{2t_\sigma}\mathcal{P}_\zeta^{-2} f \quad (173)$$

$$\langle\tau_{\rm NL}\rangle = 16t_\sigma^3\left(\frac{\Omega_\sigma}{3}\right)^4 e^{-4Nt_\sigma}\left(\frac{k}{H_0}\right)^{3t_\sigma}\mathcal{P}_\zeta^{-3} f \quad (174)$$

$$f \equiv (LH_0)^{-t_\sigma} + y(kL)t_\sigma\left(\frac{k}{H_0}\right)^{t_\sigma}. \quad (175)$$

In accordance with the discussion of Section II D these expressions are independent of the box size. But the split into the linear plus quadratic term depends on the box size. With a maximal box the linear term vanishes, even in gaussian regime $Nt_\sigma \lesssim 1$. With a minimal box the linear term dominates on large scales, even in the non-gaussian regime.

With a minimal box, negligible tilt and $\zeta = \zeta_\sigma$, Eq. (157) applies. Inserting the Bunch-Davies expectation value for σ^2 then reproduces Eqs. (166)–(168). If instead we consider the PNGB case, assuming $\Lambda \ll H_*$ so that σ has a flat distribution up to $\sigma\,{\rm max} = f$, Eq. (166) becomes

$$\mathcal{P}_\zeta^{1/2} \gtrsim \Omega_\sigma\frac{H_*}{\Lambda}\frac{\Lambda}{f} \gg \Omega_\sigma\frac{\Lambda}{f}. \quad (176)$$

Using Eq. (119) we have again $\Omega_\sigma\sqrt{t_\sigma} < 1$, making t_σ indistinguishable from 1.

G. Cosmic uncertainty

To make contact with previous work, I assume in this subsection that the curvaton field has negligible spectral tilt, and has negligible evolution before the oscillation starts. Also I consider $Q \equiv (2/5)\mathcal{P}_\zeta^{1/2}$ whose observed value is 2×10^{-5}.

We are not going to be concerned with precise values, but in this context we do not want to exclude the case $\sigma^2 \lesssim H^2$. To handle it we can replace σ in Eq. (157) by $\sqrt{\sigma^2 + H^2}$, leading to

$$Q \sim \frac{\sqrt{\sigma^2 + H^2}}{\sigma^2 + H^2 + C^2}H \sim \Omega_\sigma\frac{H}{\sqrt{\sigma^2 + H^2}} \quad (177)$$

$$f_{\rm NL}^{-1} \sim \Omega_\sigma \sim \frac{\sigma^2 + H^2}{\sigma^2 + H^2 + C^2} \quad (178)$$

In this expression, H is evaluated during inflation.

In the Bunch-Davies case, the probability distribution for σ is gaussian, making it more or less flat up to a maximum value of order the variance; $\sigma_{\rm max} \sim H/\sqrt{t_\sigma} \gg H$. In the PNGB case (Eq. (118) with $\Lambda \ll H$) the the probability distribution is almost perfectly flat, up to a smaller maximum which could be much less than H. Using the flat distribution, one can work out the non-flat distribution for the correlators. To discuss this, I take the tilt t_σ to have a small fixed value, consistent with observation.

There are three parameters C, H and σ and I will divide the parameter space into regions, separated by strong inequalities to allow simple estimates.

Consider first the case $C \ll H$. In the regime $H \ll \sigma \ll \sigma_{\rm max}$ we have $Q \simeq H/\sigma$, with Ω_σ very close to 1 and $f_{\rm NL} = -5/4$. In the regime $\sigma \ll H$, the quadratic term dominates ζ making $Q \sim 1$ in contradiction with observation. Given the flat distribution for σ, the lengths $\sigma_{\rm max} - H$ and H over which σ runs for the two cases give their relative probabilities for a randomly-located region.

Now consider the case $C \gg H$. There are three regimes;

1. The nearly gaussian regime $C \ll \sigma < \sigma_{\rm max}$. Here $Q \sim H/\sigma$ and $\Omega_\sigma \simeq 1$ making $f_{\rm NL} \simeq -5/4$.

2. The strongly non-gaussian regime $H \ll \sigma \ll C$. Here $Q \sim H\sigma/C^2$ and $f_{\rm NL}^{-1} \sim \Omega_\sigma \sim \sigma^2/C^2 \ll 1$. The lower part of this range is forbidden by observation.

3. The regime $\sigma \ll H$. Here the quadratic term of Eq. (148) dominates leading to $Q \sim H^2/C^2$, and to $f_{\rm NL} \sim 1/Q$ in contradiction with observation.

The relative probabilities, that a given region corresponds to one or other of these cases, are proportional to the intervals $\Delta\sigma$ given at the beginning of each item. Within each case, the probability distribution for Q is

$$dP = d\sigma = \frac{d\sigma}{dQ} dQ. \tag{179}$$

H. Anthropic considerations

So far we did not take into account anthropic considerations. They have gained force recently, since it appears that string theory allows a very large number of field theory lagrangians. Both the idea and the methodology of anthropic arguments are very controversial, as emphasized for instance in [5], but let us proceed.

Anthropic arguments suggest [95] that Q has to be in a range $Q_{\rm min} < Q < Q_{\rm max}$ corresponding roughly to $10^{-6} \lesssim Q \lesssim 10^{-4}$. If the cosmological constant is taken to be fixed the probability distribution within this range is more or less flat. But Weinberg argued [96] that the cosmological constant itself should be regarded as having a flat probability distribution, since there appears to be no theoretical argument that would give a definite value, in particular zero. He showed, before the data demanded it, that anthropic arguments suggest a value appreciably different from zero. As summarized in [4], subsequent studies have shown the preferred value to be compatible with observation. Accepting this viewpoint for the cosmological constant, it has been argued [97] that the probability distribution of Q within the above range is

$$dP \propto Q^3 dP_{\rm prior}, \tag{180}$$

where the prior $dP_{\rm prior}$ is the probability distribution if anthropic considerations are ignored. I will use this estimate in the following discussion, without trying to take on board the impact of some more recent work [98].

At this point, one may wonder why the focus is exclusively on the overall normalization Q^2 of the spectrum. What about the spectral tilt, and the measures $f_{\rm NL}$ and $\tau_{\rm NL}$ of non-gaussianity? The original arguments of Harrison [99] and Zeldovich [100] for tilt in the range $|t_\sigma| \lesssim 1$ may be regarded as anthropic, but we are now dealing with an observational bound more like $|t_\sigma| < 0.01$. It seems clear that no anthropic consideration will directly produce this result, and there was no objection to values $|t_\sigma| \sim 1$ before the cmb anisotropy ruled them out. Coming to non-gaussianity, it is again hard to see how anthropic consideration will directly constraint it, and there was no objection even to the extreme case $f_{\rm NL} \sim \mathcal{P}_\zeta^{-1/2}$ until it was ruled out by observation. From the anthropic viewpoint, the small tilt and non-gaussianity presumably are produced accidentally, by anthropic constraints on other parameters including Q.

If Q is generated by the inflaton perturbation in a single-component model, it depends almost entirely on some parameters in the field-theory lagrangian. (There is some dependence on the post-inflationary history via N but it would take a big variation of that history to have much effect on N.) Then the tilt also depends only on the field theory parameters, while the non-gaussianity is automatically negligible. If the field theory parameters were taken to be fixed, the prior would be a delta function and there would be no room for anthropic arguments.

In contrast, curvaton-type models depend also on the background values of one or more fields, and if inflation begins early enough one has no option but to consider their variation within the very large and smooth inflated patch that we occupy. This was pointed out some time ago in [61, 75] for the PNBG case, and has been discussed more recently in [5] for the Bunch-Davies case. The point here is that anthropic considerations concerning Q may demand, or anyhow favour, a value σ far below $\sigma_{\rm max}$.

A precisely similar situation exists with respect to the nature of the CDM. If it consists of neutralinos, or of axions created by the oscillation of cosmic strings, the CDM density is given in terms of parameters of the lagrangian. If instead it is the oscillation of a nearly homogeneous axion field which existed during inflation, the CDM density varies with our location within the inflated patch (and so does the CDM isocurvature perturbation which is inevitable in that case [17]). Linde [101] provided the first concrete realization of anthropic ideas, when he pointed out that we might need to live in a place where the axion density is untypically small. The anthropic probability for the axion density has recently been investigated [98].[12]

Now I analyse the situation for the actual curvaton model, generalizing two recent discussions [4, 5]. The spectrum Q^2 is given by Eq. (177), and we are assuming that the probability distribution is flat within a range $0 < \sigma < \sigma_{\rm max}$. There is also the the anthropic constraint $Q_{\rm min} < Q < Q_{\rm max}$. These inequalities define a rectangle in the Q-σ plane, and we can only use the part of the curve (177) that lies within this rectangle.

The location of the rectangle relative to the curve de-

[12] In this case the axion density is proportional to $(\sigma^2 + H^2)$ where σ is the average axion field in our part of the Universe and H^2 comes from the long wavelength fluctuations [17]. The authors of [98] drop the H^2 term, which might possibly affect their results.

pends on the parameters C and H which define the curve, and on $\sigma_{\rm max}$ whose value was discussed earlier. In this 3-parameter space, there will be an unviable regime where no part of the curve lies within the rectangle. In the opposite case, part of the curve is within the allowed rectangle, and putting Eq. (179) into Eq. (180) gives the probability distribution for Q;

$$dP \propto Q^3(\sigma)d\sigma \propto Q^3 \frac{d\sigma}{dQ} dQ. \tag{181}$$

Consider first the case $C \ll H$. The regime $\sigma \lesssim H$ is unviable because it gives $Q \sim 1 > Q_{\rm max}$. Therefore we consider the regime $\sigma \gg H$, and assume that $\sigma_{\rm max}$ is big enough that its value does not matter. (As $\sigma_{\rm max}$ is reduced from some such value, it remains irrelevant to the following discussion until the allowed rectangle ceases to intersect the curve making the model unviable.) This is the version of the curvaton model considered by Garriga and Vilenkin [4]. As we noted already it corresponds to $Q \simeq H/\sigma$ with Ω_σ very close to 1 and $f_{\rm NL} = -5/4$. As σ is reduced, Q rises to a maximum of order 1, but only the regime $Q < Q_{\rm max}$ is allowed. Within this regime the probability is

$$dP \propto QdQ, \tag{182}$$

making $Q = Q_{\rm max}$ the most likely value. Taking $Q_{\rm max} = 10^{-4}$, the probability that Q is at or below its observed value is $1/25$. As these authors emphasize, the low probability for the observed value need not be taken very seriously if only because $Q_{\rm max}$ is not at all well-defined.

Let us move on to the regime $C \gg H$. In this case $Q(\sigma)$ has a maximum at $\sigma \sim C$, with the value H/C. Let us consider first the case that $\sigma_{\rm max}$ is big enough for its value not to matter.

If $H/C < Q_{\rm max}$, the peak value is anthropically allowed, and because of the Q^3 factor is in fact favoured; it had better agree with observation if the anthropic argument is to work. It corresponds to $\Omega_\sigma \sim 1$ and $f_{\rm NL} \sim 1$. This is the case considered by Linde and Mukhanov [5]. Now suppose instead that H/C is bigger than $Q_{\rm max}$, so that a region around the peak is excluded. The probabilities to the right and to the left of the peak are

$$dP_{\rm right} \simeq \frac{H^3}{\sigma^3} d\sigma \tag{183}$$

$$dP_{\rm left} \simeq \left(\frac{H\sigma}{C^2}\right)^3 d\sigma. \tag{184}$$

Integrating these expressions, the the relative probability for being in the two regions are

$$\frac{P_{\rm left}}{P_{\rm right}} \sim \left(\frac{Q_{\rm max} C}{H}\right)^2 < 1. \tag{185}$$

The right-hand region is therefore preferred anthropically, leading again to the estimate (182).

We have still to consider the case that $\sigma < \sigma_{\rm max}$ is a significant constraint. As $\sigma_{\rm max}$ moves down from a large value it will at some point exclude the right hand part of the curve. Then Eq. (184) applies which gives $dP \propto Q^3 dQ$. The probability that Q is at or below the observed value is now only 1.6×10^{-3} (with $Q_{\rm max} = 10^{-4}$) which might be regarded as a catastrophe for anthropic considerations. And we are now in the regime of strong non-gaussianity, which means that depending on parameters there might be a violation of the present observational bound on $f_{\rm NL}$.

As $\sigma_{\rm max}$ moves further down, it will start to cut into the left hand part of the curve. Eventually, the peaked probability distribution that we encountered in the previous paragraph is cut off at $\hat{Q}_{\rm max} \equiv Q(\sigma_{\rm max}) < Q_{\rm max}$ so that $\hat{Q}_{\rm max}$ becomes the preferred value, which had better agree with observation if the anthropic argument is to work. Again, one has to check that $f_{\rm NL}$ is small enough.

This completes our discussion of the anthropic status of a simple version of curvaton model. We see that the situation is rather complicated. It would get still more complicated if we allowed evolution of the curvaton (not to mention the possibility that the curvaton contribution is sub-dominant) or if we considered a very non-flat prior probability for σ such as might come a departure from the probability distribution (117).

I. Comparison with a previous work

To a considerable extent Sections IV F–IV H represent a development of [5] (see also [26]). Where they are comparable, the results are in broad agreement.

An expression essentially equivalent to Eq. (165) is derived in [5, 26]. To be precise, the expressions formally coincide because our factor e^{Nt_σ} is equal to their factor (H_0^{-1}/λ_0) defined in [5]. But we make a distinction between the number of e-folds of inflation (after the observable Universe leaves the horizon) and the number of *relevant* e-folds, because it is the latter that should be identified with N.

The only other significant difference is one of interpretation, concerning the Bunch-Davies gaussian field $\sigma(\mathbf{x})$ which they call the curvaton web. One's view about the curvaton web depends on the interpretation of σ. Within the observable Universe, the local value of σ will vary from place to place, and its variation may be observable. The variation of σ might be important if, for instance, one is considering the galaxy distribution in a relatively small region surrounding our galaxy. Indeed, the average of σ within this region (at horizon exit) may be significantly different from its average within the whole observable Universe. (Analogous considerations for the axion dark matter density were pointed out for instance in [17].) With this interpretation of σ, the steep spatial gradient of $\sigma(\mathbf{x})$ evident in one of the simulations may be an observable effect, as the authors remark.

On the other hand, the analysis of local effects within the observable Universe is a rather tricky business even if one is not concerned with a varying scalar field (but in-

stead with say the local expansion rate). It may therefore be useful to have a division of labour, whereby models of the early universe give the correlators defined with the box size comfortably enclosing the whole observable Universe. These then provide the starting point for the analysis of local effects, which can be done at a later and more sophisticated stage of the research. Certainly that is the viewpoint usually taken when the curvature perturbation is supposed to come from the inflaton, and it has been the viewpoint also of the present paper. From this viewpoint, observation is sensitive to just one point on the curvaton web, whose spatial gradient ceases to be physically significant. The gaussian probability distribution for σ within the minimal box, when inserted into Eqs. (177) and (178) directly gives the cosmic uncertainty of the correlators, and with the usual 'cosmic variance' of the almost-gaussian CMB multipoles this covers all possibilities for what will be observed even though it may take some effort to work them out.

V. CURVATON-TYPE MODELS AFTER WMAP YEAR THREE

If a curvaton-type contribution ζ_σ dominates the curvature perturbation, the tensor fraction is tiny. Then the WMAP year three results [102] combined with the SDSS galaxy survey give $n-1\simeq -0.052^{+0.015}_{-0.018}$, and the result hardly changes if WMAP data are used alone or with several other relevant data sets.

If this measurement of a small negative spectral tilt holds up it has important consequences for curvaton-type models for the origin of the curvature perturbation. As was noticed in the early days of their exploration [103], the most natural expectation for these models is that the spectral tilt t_σ of the curvaton-type field σ is practically zero. This is because, in contrast with the inflaton potential in a non-hybrid model of inflation, the potential of σ does not 'know' about the end of inflation. The potential already has to be exceptionally flat just to convert the vacuum fluctuation of σ into a classical perturbation, and in the absence of any reason to the contrary one might expect that the departure from flatness will be too small to observe.

If this expectation is accepted, the spectral tilt predicted by a curvaton-type model is

$$n-1=-2\epsilon_* \qquad curvtilt0 \qquad , \tag{186}$$

where ϵ is the flatness parameter of slow-roll inflation, or more generally is the parameter $\epsilon_H\equiv -\dot{H}/H^2$. To reproduce the observed tilt we need a more or less scale-independent value $\epsilon\simeq 0.025$

Among a suite of models considered in a recent survey [104], the large-field models with $V\propto\phi^\alpha$ (chaotic inflation) give roughly the correct ϵ. The degree of tilt depends on N, defined in this context as the number of e-folds of inflation after the observable Universe leaves the horizon. The best-motivated case is $\alpha=2$, because it may be obtained as an approximation to Natural Inflation. Taking $N=50$, this gives $n-1=-0.020$ which is a bit too small compared with the observed value. The multi-component version of this potential can help by reducing $n-1$, but no investigation has been done to see how far one can go in that direction.

Increasing α increases ϵ by a factor $\alpha/2$, so that $\alpha=4$ or 6 give a tilt agreeing with observation. (If the inflaton perturbation is required to generate the curvature perturbation such values of α are excluded, but that is not the case here.) The problem with increasing α is that it lacks motivation, either from string theory or from received wisdom about field theory [104]. If one accepts an increased α it may be more sensible to regard α as a non-integer, providing an approximation to the potential over the relevant range of ϕ.

What about the possibility of giving the curvaton-like field σ a significant negative tilt t_σ, say enough to allow $n-1\simeq t_\sigma$? This requires σ to be on a concave-downward part of its potential during inflation. Taking the view that the value of σ is an initial condition to be assigned at will (possibly with anthropic restrictions) this need not be a problem. If instead the value has the stochastic probability distribution (117), there is some tension as we saw after Eq. (118) but one can still achieve the required small tilt with reasonable probability. Only in the simplest version of the actual curvaton model does Eq. (117) (with a quadratic or periodic potential) demand negligible tilt t_σ. In general then, curvaton-type models can easily give the curvaton perturbation a suitable negative tilt allowing $n-1\simeq t_\sigma$. The challenge in that case though, is to explain the actual value $n-1\simeq -0.05$ in a natural way. As was noticed a long time ago [6], a wide class of inflation models generating the curvature perturbation from the inflaton do just that, by making $n-1\sim -1/N$.

If curvaton-type models are rejected as the origin of the curvature perturbation, that rejection is itself a powerful constraint on any early-universe scenario involving scalar fields other than the inflaton. Such fields must either be heavy during inflation, or else their contribution to the curvature perturbation must be negligible. The possible problem caused by a curvaton-type contribution being too big is analogous to the moduli problem, and indeed might even be part of that problem [28, 79].

VI. CONCLUSION

The main result of Section II is the extension of [1] to include strong spectral tilt, which is allowed if the curvaton contribution gives a sub-dominant contribution to the curvature perturbation.

Section III shows how to derive Eq. (1) and its generalizations, in the presence of both spectral tilt and non-gaussianity. We note that the curvature perturbation generated by curvaton-type model long after inflation might have a very low ultra-violet cutoff $k_{\max}$. This

means that the scale of inhomogeneities in a matter component of the cosmic fluid might not extend much below the scale $10^6 M_\odot$ required for the formation of the first baryonic objects. On the other hand it might, in which case the curvaton-type contribution to ζ might be negligible on large cosmological scales, but big on sub-cosmological scales even allowing primordial black hole formation.

On the technical side, we note that the horizon-crossing formalism does not work when spectral tilt and non-gaussianity are both to be included. Instead one should evaluate the field perturbations at a fixed epoch, which might as well be the one when $k_{\rm max}$ leaves the horizon.

Section IV revisits the actual curvaton model, taking into account all of the possible effects that have been noticed. If the curvaton contribution to the curvature perturbation has strong positive tilt, it can be negligible on cosmological scales but big enough to form primordial black holes on smaller scales. Recent discussions concerning cosmic uncertainty and the anthropic status of the curvaton models are extended. This section also demonstrates that the prediction $f_{\rm NL} = -5/4$, of the simplest version of the curvaton model, can be obtained without recourse to the sudden-decay approximation. Consequently, a detection $f_{\rm NL} = -5/4$ would be a smoking gun for the simplest version of the curvaton model. Finally, in Section V we looked at the status of curvaton-type, in the light of the recent measurement of negative spectral tilt for the curvature perturbation.

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